

## Numerical Simulation of one dimensional Heat Equation: B-Spline Finite Element Method

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### Abstract

The comprehensive numerical study has been made here for the solution of One dimensional heat equation the Finite Element method is adopted for the solution with B-spline basis function the important finding of the present study is to understand the basics behind the FEM method while the B-spline basis function come into the picture here the solution is made using Quadratic B-spline and Cubic B-spline. Then for both the cases the solution is compare with analytical solution in which a good agreement is found between proposed technique. Along with that the simulation process is made using MATLAB programming in which the 3-Dimensional geometrical intrepitaion shown as a graphical presentation taken a different time steps as well as different domain of interest , the dirichlet boundary condition has been taken for the solution of this problem which is again a good achievement of the above work along with two test problem has been taken for the sake of convince. Finally the goal has been achieved in a simulation process.

**Keywords;** Quadratic B-spline, Cubic B-spline, FEM, Stability, Simulation, MATLAB

### Introduction

HEAT equation is a simple second-order partial differential equation that describes the variation temperature in a given region over a period of time. In this work, suppose the heat flows through a thin rod which is perfectly insulated along its length except at the two ends, its position in the rod is denoted as  $x$  where  $0 \leq x \leq 1$  and the length of the rod is represented as  $L$ , as illustrated in Fig. A.

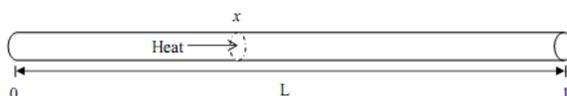


Figure A schematic diagram of temperature variation in a heated road.

$$u_t(x, t) = \beta u_{xx}(x, t) \in \Omega \times [0, T]$$

Where  $\beta$  is the thermal conductivity (1)

Where  $\Omega = [0, L]$ , with initial condition

$$u(x, 0) = u_0(x) \text{ for } 0 \leq x \leq L, \quad (2)$$

And the boundary conditions

$$u(0, t) = q_1 \text{ for } 0 \leq t \leq T, \quad (3)$$

$$u(L, t) = q_2 \text{ for } 0 \leq t \leq T, \quad (4)$$

where  $\rho, c, \kappa$  are density, thermal conductivity in the direction of  $x$ -axis and heat capacity respectively.  $u_0, f, g$  are known functions and  $u(x, t)$  is unknown. Its a well-known second order linear partial differential

equation(PDE). Unlike other equations, it is not preserved when  $t$  is replaced by  $-t$ . It shows that heat equation describes ir reversible process and makes a distance between the previous and next steps. Such equations arise very often in various applications of science and engineering describing the variation of temperature (or heat distribution) in a given region over some time. Heat equation mainly in one-dimension had been studied by many authors as in references therein [1], [2], [8], [9]. An comparative study between the traditional separation of variables method and Adomian method for heat equation had been examined by Gorguis and Benny Chan [5]. Dehghan [4] considered the use of second-order finite difference scheme to solve the two-dimensional heat equation. After that, Mohebbi and Dehghan [2] presented a fourth-order compact finite difference approximation and cubic  $C^1$ -spline collocation method for the solution with fourth-order accuracy in both space and time variables,  $O(h^4k^4)$ . In literature [7], Kumar concluded that spline gives a simple and practical method to solve the boundary problems compared to finite difference method. On the other hand, an extension of B-spline function, namely extended cubic uniform B-spline had been proposed by Han and Liu in literature [6].

In contrast to the heat equation, the steady state constant is not simply the integral of the initial condition, but also depends in some complicated way on both the viscosity parameter and the shape of the initial condition, cf., [11] in particular, numerical calculations have been used to “suggest” that Euler equations do not have unique solution, cf., [12]. The justification for this claim is that a “very fine mesh” is used in the calculation. In this note we suggest that “numerical based” proofs of non-existence must be done with extreme care. problem which arises in the quasi-static theory of thermoelasticity (Day [13], [14]). Investigators like Ekolin [15] and Liu [16] had studied the problem and introduced finite-difference methods for solving it numerically. The numerical solution of the heat equation is discussed in many textbooks. Ames [18], Morton and Mayers [20], and Cooper [17] provide a more mathematical development of finite difference methods. See Cooper [17] for modern introduction to the theory of partial differential equations along with a brief coverage of numerical methods. Fletcher [19], Golub and Ortega [21], and Homan [22] take a more applied approach that also introduces implementation issues. Fletcher provides Fortran code for several methods. Marwaha & Chopra [26] give a numerical solution for transient thermal distribution in a slab where chemical, electrical or nuclear energy is converted into thermal energy. To describe temperature profile, Elimoel and Rogerio [27] uses exponential-sinusoidal one-dimensional analytical model demonstrating that heat equation can still be solved analytically. Monte [28] applied a natural analytical approach for solving the one dimensional transient heat conduction in a composite slab. He studied the transient response of one dimensional multilayered composite conducting slabs to the sudden variations of the temperature of surrounding fluid. In a one dimensional hollow composite cylinder with time dependent boundary conditions, Lu [29] gave a novel analytical method applied to the transient heat conduction equation.

Splines are a kind of piecewise polynomials that are easy to use for the approximations by breaking an interval into a number of subintervals [30]. In general, we consider a piecewise polynomial, let  $[a, b]$  be a finite interval and  $x_0 < x_1, \dots, < x_N$  be the partition of  $[a, b]$  into  $n$  elements and  $x_i; (i = 0, 1, \dots, n)$  are the nodes. These nodes are the parameter values where the polynomial function constituting spline curve joining each other [31]. A particular class of splines are - B-splines. B-spline (bell-shaped splines) are piecewise polynomial functions that are connected continuously by the smaller curve segments that takes care of the continuity having a direct impact on the basis functions. B-spline functions are very popular for the smooth approximations and are being used by various researchers for different problems ([32], [33], [34], [35],[36]). In 1946, Schoenberg first proposed the theory of B-splines. Cox(1971) and de Boor(1972) gave recurrence relations for the purpose of computing coefficients ([37], [38]). Yang [39] used a technique to deal with the boundary conditions using splines. Many others [40], [41] has used B-splines for various applications where B-splines has been proved a useful tool for approximations. In addition to this Thomas [42] presented extensive use of splines for Boeing. Qiu [43] applied a trivariate B-spline for volume reconstruction saying that the B-spline reconstructions are often superior to the existing methods. In the present method we proposed a cubic B-spline FEM for the solution of one dimensional heat conduction equation. Because B-spline curve has superior properties making them suitable for shape representations and analysis purpose. Using the values of B-splines, nodal values at the knots are expressed in terms of the element parameters. Temperature distribution over the subintervals will be approximated by the combination of cubic B-splines and unknown element parameter. The system is reduced to the matrix form and further handled by method of lines. Simulation ,etc

**Method of Solution:** The Finite element method has been adopted for the solution for the above equation in which B-spline basis has been considered hence the technique is called as B-Spline FEM in this connection we take two different cases quadratic and Cubic which are given below two show the transpierce of the technique .

**Quadratic B-spline formulation :**

A spline is a piecewise polynomial of order  $k$  ( degree  $k-1$  at most) defined on the interval  $[a,b]$ ;  $a=x_0 < x_1, \dots, < x_{N-1} = b$  where  $h = x_{i+1} - x_i, i = 0, \dots, N - 1$ . . The spline  $u(x)$  commonly described in the B-spline representation is as  $u(x) = \sum_{i=-1}^N \alpha_i \beta_{ik}(x)$  where  $\beta_{ik}$  is a special spline function of order  $k$  called B-spline, which has a particular property of having compact support [19]. B-spline of order one are step functions defined by

$$\beta_{i1} = \begin{cases} 1; & x \in [x_i, x_{i+1}] \\ 0; & \text{otherwise} \end{cases}$$

It introduces the knots  $t_i; i = 1, \dots, N + k$ . The cubic B-spline Finite element method has been presented in this paper. The method is based on the galerkin formulation of the given problem and then using cubic B-spline basis functions. Thus the approximation to the solution is given by

$$u_N = \sum_{i=-1}^{N+1} \alpha_i(t) \beta_i(x) \tag{5}$$

where  $\beta_i$  are linearly independent B-spline basis functions and  $\alpha_i$  are the unknowns yet to be determined We start with the weak formulation of the one dimensional heat equation (1)<sub>j</sub>(4), [21] for  $f = 0, g = 0$ . Multiplying equation (1) with the test function  $N(x)$  we get

$$\int_{\Omega} N(x) \rho c \left( \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} \right) dx = 0$$

$$\int_{\Omega} (N(x) \rho c \frac{\partial u}{\partial t} + \kappa \frac{\partial N}{\partial x} \frac{\partial u}{\partial x}) dx - N(L) \kappa \frac{\partial u}{\partial x}(L, t) + N(0) \kappa \frac{\partial u}{\partial x}(0, t) = 0 \tag{6}$$

Which is the weak formulation of the given differential equation (1), whose solution will be given by the Galerkin approximation? Using (5) in (6) we have :

$$\int_{\Omega} (N(x) \rho c \left( \frac{\partial uN}{\partial t} + \kappa \frac{\partial N}{\partial x} \frac{\partial uN}{\partial x} \right) dx - N(1) \kappa \frac{\partial uN}{\partial x}(L, t) + \kappa N(0) \frac{\partial uN}{\partial x}(0, t) = 0 \tag{7}$$

using the given boundary conditions (3) - (4), we get:

$$\int_{\Omega} (N(x) \rho c \frac{\partial u}{\partial t} + \kappa \frac{\partial N}{\partial x} \frac{\partial u}{\partial x}) dx = 0 \tag{8}$$

For the Galerkin procedure, test function and the trial functions are same, thus we have  $N(x) = \mathbf{B}(x)$ . Where every test function  $N(x)$  is a linear combination of the test functions  $N(x) = \sum_{i=1}^N \alpha_i N_i(x)$  .So, the equation (8) reduces to the set of ordinary differential equations. To construct the spline approximation function for one dimensional case, the interval  $[0; 1]$  is discretized into  $(N- 1)$  intervals with  $N$  nodes.

For the spatial discrization, the domain  $\Omega$  is discrized into small intervals with  $0 = x_0 < x_1 < \dots < x_N = 1$ . Since each spline covers three intervals, for each element we take quadratic B-spline scaling function where each finite element  $[x_m, x_{m+1}]$  is covered by three quadratic B-spline  $B_j; j = -1, \dots, N$  defined by (Prenter 1975)

$$B_j(x) = \frac{1}{\kappa^2} \begin{cases} (x_{j+2} - x)^2 - 3(x_{j+1} - x)^2 + 3(x_j - x)^2, & \text{if } x \in [x_{j-1}, x] \\ (x_{j+2} - x)^2 - 3(x_{j+1} - x)^2, & \text{if } x \in [x_j, x_{j+1}] \\ (x_{j+2} - x)^2, & \text{if } x \in [x_{j+1}, x_{j+2}] \end{cases} \tag{7}$$

The length of each interval  $x_j - x_{j-1} = \kappa$  for all  $i$  and each interval is covered by three successive quadratic B-splines.  $B_j; j = -1, \dots, N$  and its first derivative vanish outside the interval  $[x_{j-1}, x_{j+2}]$ . Integrating PDE by parts we get

Approximate in term of basis function we defines

$$u_i^N(x, t) = \sum_{j=-1}^N \alpha_j(t) \beta_j(x)$$

Where  $B_j$  are quadratic B-splines basis function in given equation and  $\alpha_j$  are unknown. The approximation  $u_i^N$  and its first derivative is continuous across the solution domain. Using that the solution can be obtained using unknowns.

**Cubic B-spline formulation :**

A spline is a piecewise polynomial of order  $k$  ( degree  $k-1$  at most) defined on the interval  $[a,b]$ ;  $a=x_0 < x_1, \dots, < x_{N-1} = b$  where  $h = x_{i+1} - x_i, i = 0, \dots, N - 1$ . The spline  $u(x)$  commonly described in the B-spline representation is as  $u(x) = \sum_{i=-1}^N \alpha_i \beta_{ik}(x)$  where  $\beta_{ik}$  is a special spline function of order  $k$  called B-spline, which has a particular property of having compact support [19]. B-spline of order one are step functions defined by

$$\beta_{i1} = \begin{cases} 1; & x \in [x_i, x_{i+1}] \\ 0; & \text{otherwise} \end{cases}$$

It introduces the knots  $t_i; i = 1, \dots, N + k$ . The cubic B-spline Finite element method has been presented in this paper. The method is based on the galerkin formulation of the given problem and then using cubic B-spline basis functions. Thus the approximation to the solution is given by

$$u_N = \sum_{i=-1}^{N+1} \alpha_i(t) \beta_i(x) \tag{5}$$

where  $\beta_i$  are linearly independent B-spline basis functions and  $\alpha_i$  are the unknowns yet to be determined We start with the weak formulation of the one dimensional heat equation (1);(4), [21] for  $f = 0, g = 0$ . Multiplying equation (1) with the test function  $N(x)$  we get

$$\int_{\Omega} N(x) \rho c \left( \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} \right) dx = 0$$

$$\int_{\Omega} (N(x) \rho c \left( \frac{\partial u}{\partial t} + \kappa \frac{\partial N}{\partial x} \frac{\partial u}{\partial x} \right) dx - N(L) \kappa \frac{\partial u}{\partial x}(L, t) + N(0) \kappa \frac{\partial u}{\partial x}(0, t) = 0 \tag{6}$$

Which is the weak formulation of the given differential equation (1), whose solution will be given by the Galerkin approximation? Using (5) in (6) we have :

$$\int_{\Omega} (N(x) \rho c \left( \frac{\partial uN}{\partial t} + \kappa \frac{\partial N}{\partial x} \frac{\partial uN}{\partial x} \right) dx - N(1) \kappa \frac{\partial uN}{\partial x}(L, t) + \kappa N(0) \frac{\partial uN}{\partial x}(0, t) = 0 \tag{7}$$

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Let  $[a, b]$  be partitioned into a mesh by points  $x_1, x_2, \dots, x_N$  such that  $a = x_0 < x_1, \dots, < x_{N-1} < x_N = b$ . Consider the knots  $x_i; x_i = a + ih, h = \frac{b-a}{N}$ . The cubic B-spline covering four successive

finite elements, are covered by the given four shape functions. Thus variation of  $u$  over each element  $[x_m, x_{m+1}]$  is expressed as

$$u(x, t) = \sum_{j=m-1}^{m+2} \beta_j(x_i) \delta_j(t), i = 0, 1, \dots, N \tag{9}$$

where  $\delta_j(t)$  are element parameters and  $\beta_j; (j = m - 1, m, m + 1, m + 2)$  are element shape functions. The cubic B-spline  $\beta_j(x)$  at the nodes  $x_i$  are defined to form a basis over the given interval

$$B_j = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & \text{if } x \in [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3 & \text{if } x \in [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3 & \text{if } x \in [x_{m-2}, x_{m-1}] \\ (x_{m+2} - x)^3 & \text{if } x \in [x_{m-2}, x_{m-1}] \\ 0 & \text{otherwise} \end{cases}$$

Where  $j = -1, 0, \dots, N + 1$ . Each finite element is defined on the interval  $[x_j, x_{j+1}]$  and the element nodes are defined for the knots  $x_j, x_{j+1}$ . Using (9)&(10) as weight functions in (8), we get

$$\sum_{j=m-1}^{m+2} (\rho c \int_{x_m}^{x_{m+1}} \beta_i \beta_j dx) \delta_j + \sum_{j=m-1}^{m+2} (\kappa \int_{x_m}^{x_{m+1}} \beta_i' \beta_j' dx) \delta_j \tag{11}$$

For each element we can write the matrix form

$$M^e \frac{d\delta^e}{dt} + K^e \delta^e = 0 \tag{12}$$

Where  $\delta = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$  are the elements parameters and  $M^e, K^e$  are elements matrices given by

$M_{ij}^e = \rho c \int_{x_m}^{x_{m+1}} \beta_i \beta_j dx, K_{ij}^e = \kappa \int_{x_m}^{x_{m+1}} \beta_i' \beta_j' dx$ , where  $i, j = m - 1, m, m + 1, m + 2$  and the element matrices obtained as

$x$	$x_{j-2}$	$x_{j-1}$	$x_j$	$x_{j+1}$	$x_{j+2}$
$\beta_j(x)$	0	1	4	1	0
$\beta_j'(x)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0

By assembling together contributions from all the elements, we get the matrix equation

As

$$M \frac{d\delta}{dt} + K \delta = 0 \tag{13}$$

where  $M, K$  are assembled form of the matrices  $M^e$  &  $K^e$  respectively. Using linear variation of time interval,  $t^0 < t^1 < \dots < t^n < t^{n+1}$  ...be the partition of the temporal domain. Defining  $\Delta t = t^{n+1} - t^n$  as the length of the time interval.  $\Delta t = \lambda$ , From Taylor series, temperature at time level  $(n + 1)$  is

$$\delta^{n+1} = \delta^n + \lambda \frac{\partial \delta^n}{\partial t} + \frac{\lambda^2}{2} \frac{\partial^2 \delta^n}{\partial t^2} + \dots$$

$$\frac{\partial \delta^n}{\partial t} \approx \frac{\delta^{n+1} - \delta^n}{\lambda} + o(\lambda) \tag{14}$$

We can now use a numerical method to integrate the system obtained, so called the method of lines. To solve (13), we use  $\theta$  - family of approximation where  $0 \leq \theta \leq 1$  weight factor such that

$$\delta^{n+\theta} = \theta\delta^{n+1} + (1 - \theta)\delta^n \quad (15)$$

Using (14) and (15) into (13) we have

$$[M] \left( \frac{\delta^{n+1} - \delta^n}{\lambda} \right) + [K]\delta^{n+\theta} = 0$$

$$[M] \left( \frac{\delta^{n+1} - \delta^n}{\lambda} \right) + [K](\theta\delta^{n+1} + (1 - \theta)\delta^n) = 0$$

A simple rearrangement given us

$$(M + \theta\lambda K)\delta^{n+1} = (M - \lambda(1 - \theta)K)\delta^n \quad (16)$$

Choosing  $\theta = 1/2$  corresponds Crank-Nicolson method for solving (16). So, we have

$$\delta^{n+1}(2M + K\lambda) = \delta^n(2M - K\lambda) \quad (17)$$

which can be solved for each time step  $\Delta t$  to find the approximate solution of (1)-(4)

### Numerical Result

For the numerical solution of one dimensional heat conduction equation using the above technique we consider two

**Experiment problem (1)** For equation (1) we consider an iron bar of length 50cm with specific heat =  $0.437 J/(gK)$ , density  $\rho = 7.88 g/cm^3$  and thermal conductivity  $k = 0.836 W/(cmK)$  subject to the following initial

$$u(x, 0) = 5 - \frac{1}{5}|x - 25| \text{ for } 0 \leq x \leq 50 \quad (19)$$

With the boundary condition

$$u(0, t) = u(50, t) \quad (20)$$

The analytical solution to this problem can be expressed as

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{50}\right) \quad (21)$$

Where

$$A_n = \frac{40 \sin\left(\frac{n\pi}{2}\right)}{\pi^2 n^2} e^{-\frac{kn^2\pi^2 t}{2500\rho c}}$$

**Experiment Problem (2)** An iron bar of length 50cm being heated at a constant temperature  $4^\circ C$  and both the ends are maintained at  $0^\circ C$  and  $4^\circ C$  respectively. Mathematical modeling of the given phenomenon is represented in the form of governing PDE with the initial condition as  $u_0 = 4^\circ C$  and boundary condition  $u(0, t) = 0^\circ C$ ,  $u(50, t) = 4^\circ C$ . whose analytical solution is given by

$$u = \frac{4}{50}x + \sum_{n=1}^{\infty} \frac{8}{n\pi} \exp\left(-\frac{kn^2\pi^2 t}{\rho c 50^2}\right) \sin\left(\frac{n\pi x}{50}\right)$$

The proposed techniques leads to the final matrix formulation which is solved using Crank-Nicolson technique. To observe the accuracy of the numerical technique, weighted  $\|e\|_1$  norm is used defined by

$$\|e\|_1 = \frac{1}{N} \sum_{i=1}^{N-1} \left| \frac{U(x_i, t_j) - U_{i,j}}{U(x_i, t_j)} \right|, e = [e_1, e_2, e_3 \dots \dots, e_{N-1}]^T$$

Table 1 shows the numerical results of case 1 for different meshes. Numerical results have been found in a very good agreement with the exact solution. For different materials with conductivities, temperature variation is shown in Fig. 1 at different times  $t = 0.04, 0.1, 0.25, 0.5$ . For case 2 numerical solutions are given in Table 2. Results obtained from the present (B-spline FEM) technique are compared with the exact solutions with  $\Delta t = 0.0001 \Delta x = 0.05$  at different times. It gives the numerical results very much closer to the respective exact solutions. In Fig. (2) temperature profile for case (2) is studied at different times  $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ . At the initial level (Initial condition) temperature is  $4^\circ$ . When the effect of boundary condition. When the effect of boundary condition come into picture, the profile is representing a semi parabolic shape at  $t=0.1$ . The temperature at left side is  $0^\circ c$  and reached other side at  $4^\circ c$ . With the passage of time, shape is being changes to the inclined line. No further variation in time is noticed after  $t = 0.8$  and temperature profile reaches the steady state.

**Numerical simulation using MATLAB:** A generalized simulation process has been shown here for the FEM technique with the B-spline basis functions, our objective is to understand the implementation of the B-spline basis function while solving the differential equation using MATLAB it is a very important part of our study, so exactly the code of the problem can not shown here, but a simple algorithm corresponding to a simple problem being shown here to understand the basics of computational process ::

Let us consider an ordinary differential equation of practical problem, to find out the launch velocity of the rocket say  $y'(0)$

$$\frac{d^2y}{dx^2} = c,$$

- $c=9.6$  [m/sec<sup>2</sup>] (acceleration due to gravity)
- $y(0)=0$  (launch from ground)
- $y(5)=40$  (fireworks explode after 5 seconds, we want them 40 m off ground)

IMP: This can be solved analytically; exact motion is quadratic in  $t$ , final answer is  $y'(0)=32.5$  [m/sec].

#### MATLAB Programme:

```
function galerkin(n)
% Galerkin finite element method to solve fireworks problem
% n is number of divisions
% using quadratic B-spline basis functions

global g te ye h;

c = 9.8;           % accel. of gravity, [m/sec^2]
te = 5;           % end time [sec]
ye = 40;          % end height [m]

h = te/n;

A(1, 1:2) = [.5 .5];
b(1) = 0;          % y=0 at t=0

A(2, 1:4) = [4 -7 2 1]; % Galerkin condition on first interior
intv1
b(2) = -5*g*h^2;

for i = 3:n
    A(i, i-2:i+2) = [1 2 -6 2 1];
    b(i) = -6*g*h^2;
end

A(n+1, n-1:n+2) = [1 2 -7 4]; % last interior interval
b(n+1) = -5*g*h^2;
```

```

A(n+2, n+1:n+2) = [.5 .5];
b(n+2) = ye; % y=ye at t=te

b = b'; % make b into a column vector

u = A\b;
A
b'
u'

(u(2)-u(1))/h % initial velocity

% plot(-h/2:h:te+h/2, u, 'g'); % plot the coefficients

tv = 0:h/8:te; % supersample the parabolas
yv = [];
for t = tv
    i = floor(t/h)+2;
    if i==n+2
        i = n+1;
    end
    x = t/h-i+1.5;
    y = u(i-1)*.5*(x-.5)^2 + u(i)*(.75-x^2) + u(i+1)*.5*(x+.5)^2;
    yv = [yv y];
end

% compute joints between elements
tk = (0:h:te)';
yk = .5*(u(1:n+1)+u(2:n+2));

hold off;
set(gca, 'FontSize', 16); % for tick marks
plot(tv,yv,'b', tk,yk,'bo', 'LineWidth',2);
xlabel('t', 'FontSize',20);
ylabel('y', 'FontSize',20);
axis([0 5 0 60]);
title('Galerkin Finite Element Method', 'FontSize',20);

```

Here is what the above code prints out.

#### **galerkin.out**

```

>> galerkin(4)
A =
    0.5000    0.5000         0         0         0         0
    4.0000   -7.0000    2.0000    1.0000         0         0
    1.0000    2.0000   -6.0000    2.0000    1.0000         0
         0    1.0000    2.0000   -6.0000    2.0000    1.0000
         0         0    1.0000    2.0000   -7.0000    4.0000
         0         0         0         0    0.5000    0.5000

b =
         0   -76.5625   -91.8750   -91.8750   -76.5625   40.0000

u =
   -20.3125   20.3125   45.6250   55.6250   50.3125   29.6875

ans =
    32.5000

```

**Table:1A** Comparison of numerical results for various mesh sizes at time  $t = 0.1$  for  $\Delta t = 0.0001$ , when Quadratic B-Spline has been taken into consideration

x	h=0.01	h=0.05	h=0.0125	h=0.0075	Exact
0.2	0.386302345	0.386214579	0.386192865	0.386154367	0.386113889
0.4	0.798894907	0.798800034	0.798798426	0.798723452	0.798795288
0.6	1.236846899	1.214511564	1.214609281	1.214600533	1.214664441
0.8	1.602508693	1.602537398	1.602528645	1.602376544	1.602562236
1.0	1.983713468	1.983603475	1.983004267	1.983005526	1.983493017
1.2	2.395822888	2.395747239	2.395787667	2.395710576	2.395735181
1.4	2.820016994	2.820165748	2.820123484	2.820263489	2.820284157
1.6	3.206353967	3.206454224	3.206975337	3.206546582	3.206541098
1.8	3.571600496	3.571437654	3.571273612	3.571987654	3.571225499
2.0	3.991578296	3.991350098	3.991675342	3.991168087	3.991118273
2.2	4.453106093	4.454347638	4.454826787	4.454987654	4.454725335
2.4	4.782141265	4.783280500	4.783992693	4.784346769	4.784262902
2.6	4.782187204	4.783213234	4.783369276	4.784134312	4.784262902
2.8	4.453342167	4.410865478	4.454926599	4.454984325	4.454725335
3.0	3.991884088	3.990002644	3.991162979	3.991132345	3.991118273
3.2	3.571657830	3.571864087	3.571223457	3.571254374	3.571225499
3.4	3.206473211	3.206297693	3.206598324	3.206543467	3.206541098
3.6	2.820000456	2.820175689	2.820204300	2.820345477	2.820284157
3.8	2.395432167	2.395386769	2.395745634	2.395742532	2.395735181
4.0	1.983719849	1.989050407	1.983534798	1.983456564	1.983493017
4.2	1.602558863	1.602523789	1.602553233	1.602313468	1.602562236
4.4	1.214400003	1.214519453	1.214876123	1.214649063	1.214664441
4.6	0.798698762	0.798629056	0.798792986	0.798754322	0.798795288
4.8	0.386351275	0.386275752	0.386784567	0.386254445	0.386113889
$\ e\ _1$	0.00031	0.0000077766	0.00000195	0.00000117	

**Table:1B** Comparison of numerical results for various mesh sizes at time  $t = 0.1$  for  $\Delta t = 0.0001$ , when Cubic B-Spline has been taken into consideration

x	h=0.01	h=0.05	h=0.0125	h=0.0075	Exact
0.2	0.386307933	0.386211243	0.386138291	0.386128534	0.386113889
0.4	0.798820192	0.798807796	0.798798426	0.798797171	0.798795288
0.6	1.214460634	1.214562193	1.214638814	1.214649063	1.214664441
0.8	1.602507627	1.602534805	1.602555354	1.602558105	1.602562236
1.0	1.983719849	1.983606803	1.983521533	1.983510132	1.983493017
1.2	2.395832333	2.395783988	2.395747426	2.395742532	2.395735181
1.4	2.820010128	2.820146727	2.820249721	2.820263489	2.820284157
1.6	3.206368202	3.206454224	3.206519297	3.206528011	3.206541098
1.8	3.571608764	3.571417599	3.571273612	3.571254374	3.571225499
2.0	3.991483713	3.991301987	3.991164388	3.991145957	3.991118273
2.2	4.453926101	4.454325872	4.454625498	4.454665435	4.454725335
2.4	4.782149974	4.783205327	4.783998299	4.784104124	4.784262902
2.6	4.782149974	4.783205327	4.783998299	4.784104124	4.784262902
2.8	4.453926101	4.454325871	4.454625498	4.454665435	4.454725335
3.0	3.991483713	3.991301987	3.991164388	3.991145957	3.991118273
3.2	3.571608764	3.571417599	3.571273612	3.571254374	3.571225499
3.4	3.206368202	3.206454228	3.206519297	3.206528011	3.206541098
3.6	2.820010128	2.820146727	2.820249721	2.820263489	2.820284157
3.8	2.395832333	2.395783988	2.395747426	2.395742532	2.395735181
4.0	1.983719849	1.983606803	1.983521533	1.983510132	1.983493017
4.2	1.602507627	1.602534805	1.602555354	1.602558105	1.602562236
4.4	1.214460634	1.214562193	1.214638814	1.214649063	1.214664441
4.6	0.798820192	0.798807796	0.798798426	0.798797171	0.798795288
4.8	0.386307933	0.386211243	0.386138291	0.386128534	0.386113889
$\ e\ _1$	0.00031	0.0000077766	0.00000195	0.00000117	

**Table:2A** For  $\Delta t = 0.0001, \Delta x = 0.05$ , Comparison of numerical results at different times, when Quadratic B-Spline has been taken into consideration

x	h=0.01	h=0.05	h=0.0125	h=0.0075	Exact
02.0	0.306493	0.369876	0.304695	0.308754	0.304721
04.0	0.604245	0.601234	0.600987	0.600987	0.606671
06.0	0.902789	0.909536	0.901237	0.903109	0.903157
08.0	1.192679	1.198762	1.199876	1.193456	1.191636
10.0	1.469834	1.466790	1.465678	1.469876	1.469764
12.0	1.730053	1.730568	1.733456	1.734345	1.735493
14.0	1.982733	1.912670	1.986929	1.989876	1.987064
16.0	2.222863	2.265214	2.220987	2.229876	2.223073
18.0	2.456872	2.407548	2.445679	2.412678	2.442567
20.0	2.644378	2.649367	2.649863	2.698423	2.644663
22.0	2.822983	2.845345	2.826456	2.898737	2.829265
24.0	2.996383	2.912945	2.934567	2.991230	2.996348
26.0	3.146398	3.140920	3.148765	3.145679	3.146254
28.0	3.273983	3.272872	3.245689	3.271456	3.279638
30.0	3.338009	3.321380	3.399735	3.390589	3.397349
32.0	3.507738	3.502934	3.509843	3.508643	3.500473
34.0	3.594569	3.592097	3.512349	3.523457	3.590235
36.0	3.660679	3.623878	3.698467	3.660876	3.667985
38.0	3.731145	3.730083	3.738767	3.730740	3.735159
40.0	3.793699	3.797744	3.799854	3.793687	3.793283
42.0	3.846469	3.843349	3.840998	3.848437	3.843663
44.0	3.887991	3.881234	3.823423	3.823480	3.888052
46.0	3.924567	3.922934	3.921234	3.920654	3.927891
48.0	3.960987	3.962178	3.969876	3.969532	3.964768
$\ e\ _1$	0.00013832	6.09296e-05	4.57977e-05	3.07261e-05	

**Table:2B** For  $\Delta t = 0.0001, \Delta x = 0.05$ , Comparison of numerical results at different times when Cubic B-Spline has been taken into consideration

x	h=0.01	h=0.05	h=0.0125	h=0.0075	Exact
02.0	0.304685	0.304687	0.304695	0.304704	0.304721
04.0	0.606601	0.606605	0.606621	0.606638	0.606671
06.0	0.903055	0.903062	0.903084	0.903109	0.903157
08.0	1.191511	1.191517	1.191547	1.191574	1.191636
10.0	1.469613	1.469615	1.469655	1.469691	1.469764
12.0	1.735313	1.735324	1.735363	1.735475	1.735493
14.0	1.986861	1.986871	1.986929	1.986974	1.987064
16.0	2.222863	2.222873	2.222923	2.222973	2.223073
18.0	2.442271	2.442291	2.442342	2.442392	2.442567
20.0	2.644431	2.644451	2.644512	2.644557	2.644663
22.0	2.829044	2.829055	2.829131	2.829153	2.829265
24.0	2.996124	2.996139	2.996184	2.996233	2.996348
26.0	3.146043	3.146054	3.146197	3.146159	3.146254
28.0	3.279422	3.279441	3.279484	3.279532	3.279638
30.0	3.397151	3.397161	3.397218	3.397256	3.397349
32.0	3.500345	3.500318	3.500353	3.500391	3.500473
34.0	3.590083	3.590091	3.590128	3.590164	3.590235
36.0	3.667854	3.667856	3.667894	3.667927	3.667985
38.0	3.735032	3.735034	3.735062	3.735091	3.735159
40.0	3.793131	3.793119	3.793134	3.793155	3.793283
42.0	3.843583	3.843593	3.843614	3.843627	3.843663
44.0	3.887991	3.887995	3.888012	3.888024	3.888052
46.0	3.927851	3.927853	3.927865	3.927875	3.927891
48.0	3.964681	3.964687	3.964691	3.964692	3.964768
$\ e\ _1$	0.00013832	6.09296e-05	4.57977e-05	3.07261e-05	

**Geometrical Interpretation:**

The three dimensional geometrical Interpretation of the above phenomena has been shown in the Fig 1 to 5. for different time level  $t=0.1, 0.2$  respectively for the distinguish coordinate domain, there is some duplication is seemed at different time level.

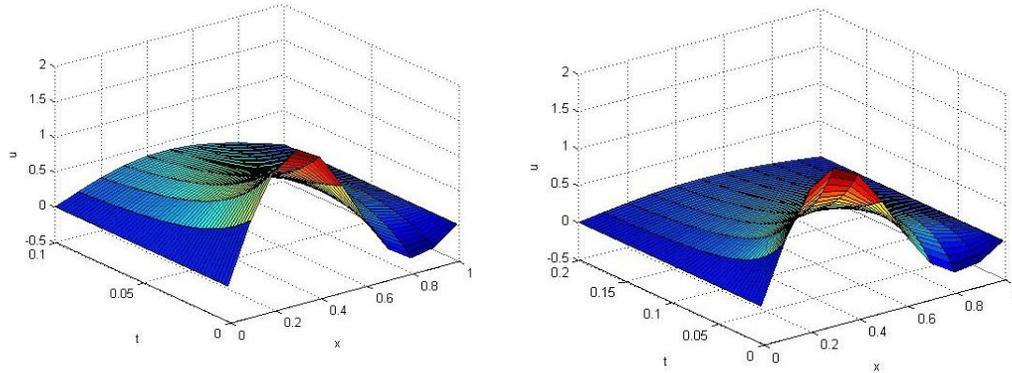


Fig 1(A) and 1 (B)

Figure 1.The geometrical Interpretation of Heat equation at different time level  $t=0.1$  and  $0.2$  at  $0 < x < 1$

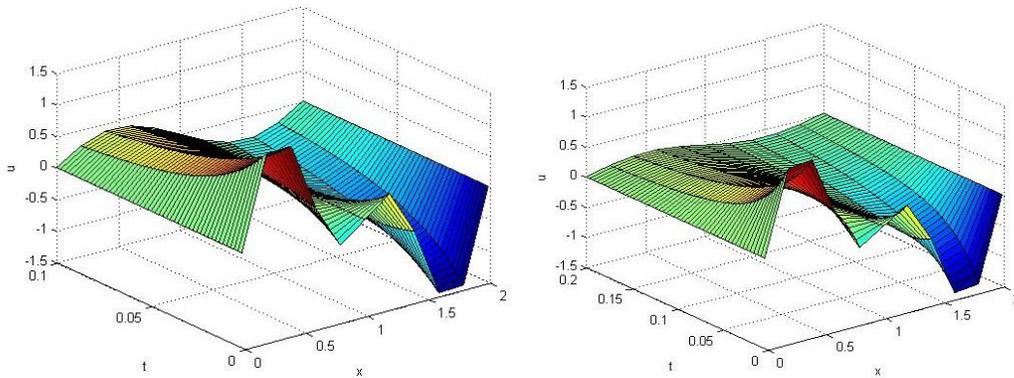


Fig 2(A) and 2 (B)

Figure 2.The geometrical Interpretation of Heat equation at different time level  $t=0.1$  and  $0.2$  at  $0 < x < 2$

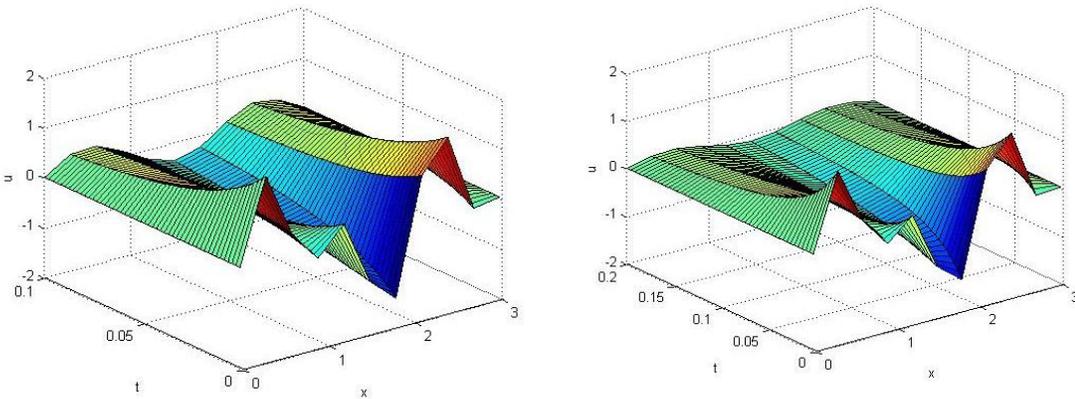


Fig 3(A) and 3 (B)

Figure 3.The geometrical Interpretation of Heat equation at different time level  $t=0.1$  and  $0.2$  at  $0 < x < 3$

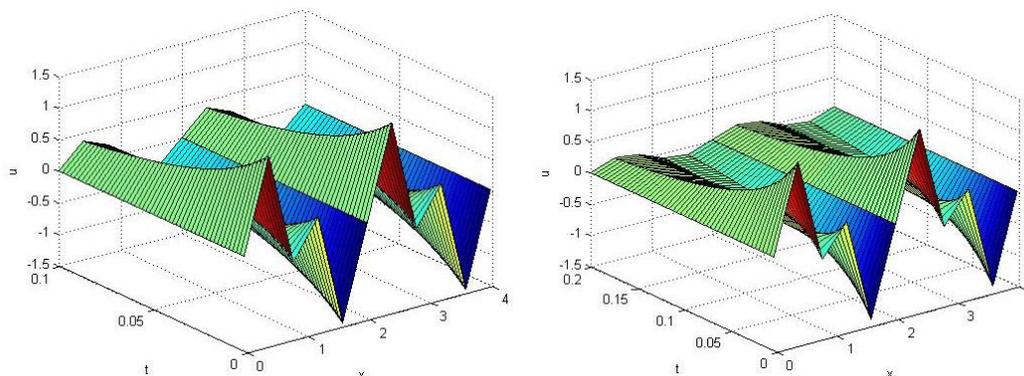


Fig 4(A) and 4 (B)

Figure 4.The geometrical interpretation of Heat equation at different time level  $t=0.1$  and  $0.2$  at  $0 < x < 4$

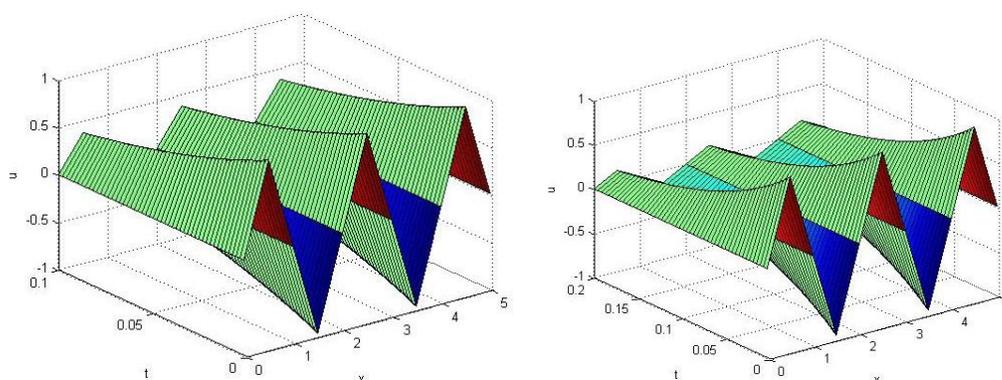


Fig 5 (A) and 5 (B)

Figure 5.The geometrical Interpretation of Heat equation at different time level  $t=0.1$  and  $0.2$  at  $0 < x < 5$

## Conclusion

In the present study we found the remarkable accuracy in the proposed technique while table-1,2 is taken in to consideration w.r.t the analytical solution and quadratic as well as cubic B-Spline function has to be taken along with that successfully 3-D geometry has to achieved in this manuscript. Furthermore, Temperature variation studied using the quadratic and cubic B-spline FEM in this paper, gives satisfactory results. Both the case studies gave sufficiently good agreements with the exact solutions. Algorithms are developed to evaluate B-spline basis functions as well as to use them for the evaluation of temperature distribution. Performance of the algorithms has been investigated by means of comparison with the analytical solutions and weighted  $\|e\|_1$  norm. It can be said that the method is good enough to study the temperature distribution in one dimension. For the future work we aim attending numerical solutions for the comparatively complex problems using the similar technique.

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