A brief introduction to Combinatorial Game Theory through the analysis of the impartial perfect information game – Nim

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Abstract

Combinatorial game theory is an upcoming field with wide applications in areas of mathematics and computer science. In this paper, we try to define a combinatorial game. Then we show a very simple example of one such game. Finally we discuss about Nim that belongs to a special class of games known as impartial games and we look at the winning strategies for that game.

Keywords

Combinatorial games, Nim, Nim-sum, two-player games, impartial games

1. Introduction

The field of combinatorial game theory is relatively new. It came into prominence only after the publication to two seminal works in this field – On Number and Games by J.H. Conway [1] and Winning Ways by Berlekamp, Conway and Guy [2]. Combinatorial games are a very specific class of games having certain well-defined characteristics. Let us enlist those game attributes that make a game combinatorial.

i. **Two players** is a mandate for combinatorial games.
ii. There should be **no chance** element involved (like rolling of dice or shuffling of cards).
iii. Both players must have **perfect information** at all times. This means everything related to the game is completely laid out on the board for both players to see at all times.
iv. Both players must move by taking turns one at a time. This **turn based** nature ensures speed is not a factor in analysis.
v. The game must eventually end after finite number of moves. Also there has to be an **absolute winner**; the game cannot end in a draw.

During normal play, the last player to move is the winner. However in **misère** variant, last player to move is the loser.

Another assumption made during analysis of such games is that all players play optimally. In reality, though, it is common for a player to make sub-optimal moves due to complexity of the game.

2. Classification

All games under combinatorial game theory can be classified into two categories –**impartial** games and **partisan** games.

- Impartial games – where the set of moves available from any given position is same for both players. We'll look into a two player impartial game in detail in a bit. You can find more about these impartial combinatorial games in Fair Game by Richard K. Guy [3].
- Partisan games – where each player has a different set of moves from a given position. Games like chess, checkers etc. comes under this category.

Following diagram gives a clear idea about the relationship between all finite games.

Fig 1: Set of all finite games

3. A Simple Game

Before going into formal discussion of impartial games let us look at a very simple game of removing coins. Suppose you have a pile of 23 identical coins and two players A and B taking alternate turns to make a move (A moving first). A move consists of removing 1, 2 or 3 coins from the pile at one go. The player to remove the last coin wins.
Looking at the above game there arises many questions. Does any player have advantage over the other? Which is a better position, playing first or second? What is a good strategy to follow?
We'll try to analyze the game from end to the beginning. This approach is known as **backward induction**.
If there are one, two or three coins left, the player to move next wins simply by taking all the chips. Suppose four coins are left. Then the player who moves next can leave either one, two or three coins for his opponent making him the winner. Thus four coins left for the next player is a winning position for the previous player.
Let us tabulate all the position with the assumption that player to move next is B.

Number of coins left by A	Winner
1, 2 or 3	B
4	A
5, 6 or 7	B
8	A
9, 10 or 11	B
12	A
13, 14 or 15	B
16	A
17, 18 or 19	B
20	A
21 or 22	B

Fig 2: Winning Position tabulation starting with 23 coins

From the above table it is clear that if A leaves 22 or 21 coins for B (i.e. A picks 1 or 2 coins at the beginning) then B wins playing optimally. Thus strategy for winning this game is leaving 4n number of coins for your opponent. If initial size of pile is multiple of 4 then player to move second will win otherwise player to move first wins, both playing optimally.

4. The classic combinatorial game – Nim

Although there are many variations of Nim, we'll stick with the following rules for our analysis. Nim is a two player game. The players must decide who moves first before start. The board consists of groups of piles of identical things (like coins or matchsticks). Usually the piles vary in sizes. A player moves by removing any number of coins from a single pile on the board. A player cannot take coins from multiple piles in the same turn. Also a player must take at least one coin every turn. The player to remove the last coin or pile of coins is the winner.

A sample Nim board looks like this.

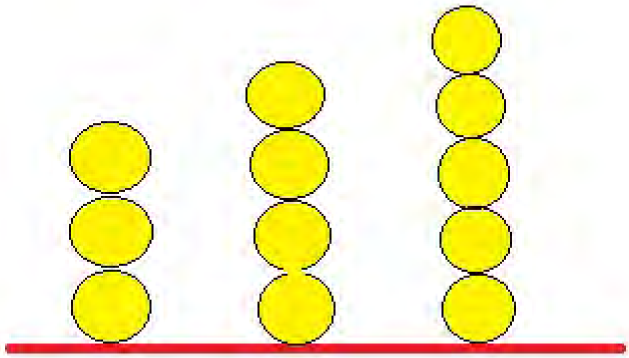

Fig 3: Simple Nim board with 3 piles

As we can see there are 3 piles with 3, 4 and 5 coins respectively. We'll use the notation (x, y, z) to represent the state of the board at any given time. Thus starting board is represented by (3, 4, 5). Like before, two players A and B take alternate turns at making moves with A moving first. Let us try to solve this game and find the winning strategy for both A and B.

5. Solution of Nim

Here we'll discuss the C.L. Bouton's solution [4] proposed in 1902. In that paper Bouton introduced the concept of Nim-sum.

- **Nim-sum**: The nim-sum of two non-negative integers is their addition without carry in base 2. Nim-sum is associative as well as commutative.
  Any number x can be expressed in base 2 in the following way
  $x_m 2^m + x_{m-1} 2^{m-1} + \cdots + x_0 2^0$ for some $\boldsymbol{m}$, where each $\boldsymbol{x_i}$ is either 0 or 1. Like 22 = $(10110)_2$
  Coming back to nim-sum; let us take two integers, say 22 and 15. Now to find their nim-sum we proceed as follows. First we express them as binary numbers, and then perform bitwise addition without carry to get the nim-sum.

  $22 = 10110_2$

  $15 = 1111_2$

  Nim-sum = $11001_2 = 25$

  Thus nim-sum of 22 and 15 is 25.

Bouton states in his paper [4] that if nim-sum of all the pile sizes in starting configuration is non-zero then first player to move has a winning strategy. And the strategy is to make the nim-sum of the resulting configuration zero. If the board has zero nim-sum to start with then second player has winning strategy.

Now let us apply the above finding to our sample game with starting configuration (3, 4, 5). Now nim-sum (3, 4, 5) = 2. Since, initial nim-sum is non-zero; player A has a winning strategy. A's strategy should be to make resulting nim-sum as zero. Only way it can be done is by picking 2 coins from first pile. Thus resulting board becomes (1, 4, 5) and nim-sum (1, 4, 5) = 0. Now whatever move B makes, the nim-sum will become non-zero nevertheless.

Let us now tabulate the scenario.

Board configuration	Nim-sum	Move
(3, 4, 5)	2	
		A takes 2 coins from first pile
(1, 4, 5)	0	
		B takes 1 coin from first pile
(0, 4, 5)	1	
		A takes 1 coin from third pile
(0, 4, 4)	0	
		B takes 3 coins from second pile
(0, 1, 4)	5	
		A takes 3 coins from third pile
(0, 1, 1)	0	
		B takes 1 coin from third pile
(0, 1, 0)	1	
		A takes last coin and wins

Fig 4: Analysis of Nim with starting configuration (3, 4, 5)

Thus we can see that following the winning strategy from the beginning, player A is able to make the last move and win the game.

6. Conclusion

In this paper, we have given a basic introduction to combinatorial game theory with special focus on impartial games. Combinatorial game theory (CGT) is an emerging field that has generated tremendous interests among mathematicians and computer scientists. CGT is applied to a wide range of areas in mathematics and computer science like alpha-beta pruning in search algorithms and multi-agent systems (MAS).

7. References

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