SUPER STRONGLY PERFECT NESS OF SOME INTERCONNECTION NETWORKS

R. Mary Jeya Jothi
Research Scholar, Department of Mathematics, Sathyabama University, Chennai,
Tamilnadu, 600119, India,
jeyajothi31@gmail.com.

A. Amutha
Assistant Professor, Department of Mathematics, Sathyabama University, Chennai,
Tamilnadu, 600119, India,
amudhajo@gmail.com.

Abstract

A Graph G is Super Strongly Perfect Graph if every induced sub graph H of G possesses a minimal dominating set that meets all the maximal complete sub graphs of H. In this paper we have analyzed the structure of super strongly perfect graphs in some Interconnection Networks, like Mesh, Torus, Hyper cubes and Grid Networks. Along with this investigation, we have characterized the Super Strongly Perfect ness in Mesh, Torus, Hyper cubes and Grid Networks. Also we have given the relationship between diameter, domination and co - domination numbers of Mesh, Torus, Hyper cubes and Grid Networks.

***Keywords*: Super Strongly Perfect Graph, Minimal Dominating Set, Mesh, Torus, Hyper cubes and Grid Networks.**

1. Introduction

Graph theoretical concepts are widely used to study and model various applications, in different areas. They include, study of molecules, construction of bonds in chemistry and the study of atoms, sociology, biology and conservation efforts (where a vertex represents regions where certain species exist and the edges represent migration path or movement between the regions). The major role of graph theory in computer applications is the development of graph algorithms. Numerous algorithms are used to solve problems that are modeled in the form of graphs [Shrinivas, et. al. (2010)]. Also, graph theory is used in research areas of computer science such as data mining, image segmentation, clustering, image capturing, networking etc., Graph theoretical concepts are widely used in Operations Research (the travelling salesman problem, problems of scheduling, etc.,), Networks (PERT - Project Evaluation Review Technique, CPM - Critical Path Method).

An interconnection network plays a central role in determining the overall performance of a multicomputer system. If the network cannot provide adequate performance, for a particular application, nodes will frequently be forced to wait for data to arrive. Some of the most important networks include Mesh, Rings, Hypercube, Butterfly, Benes and Cube Connected Cycles etc., [Xiao and Parhami (2005)]. The interconnection network connects the processors of a parallel and distributed system. The topology of an interconnection network for a parallel and distributed system can always be represented by a graph, where each vertex represents a processor and each edge represents a vertex - to - vertex communication link. Communication is a critical issue in the design of a parallel and distributed system [Huang, et. al. (2008)].

The Mesh network is a local area network (LAN) that employs one of two connection arrangements, full mesh topology or partial mesh topology. Mesh Networks are used in Wireless mesh network, Distinct radio node deployments of Wireless Mesh Networking, BioWeb, Wireless ad hoc network, Wireless community network, Mobile ad hoc network MANET), Vehicular ad-hoc network, Intelligent Vehicular AdHoc Network and etc., A Torus network has become increasingly important to multicomputer design because of its many features including scalability, low bandwidth and fixed degree of nodes. A multicast communication is a significant operation in multicomputer systems and can be used to support several other collective communication operations. The mesh and torus networks have been recognized as versatile interconnection networks for massively parallel computing [Bilardi and Preprata (1995)].

It is well known that the *n*-dimensional hypercube, denoted by Q_n, is one of the most popular and efficient interconnection networks. The n - dimensional hypercube is particularly compact. This logical structure is extremely useful because of the wide range of algorithms that fit it particularly well. It possesses many excellent properties such as recursive structure, symmetry, low degree, popular structure embedding, and easy routing. The Hypercube interconnection network is a multidimensional binary cube with a processor or processors cluster at each of its vertices [Harary, et, al (1988)]. There is a large amount of literature on graph-theoretical properties of hyper cubes and their applications in parallel computing, hypercube is widely used in early multicomputers, has fallen out of favour to be replaced by the 2 -dimensional mesh or torus in recent multicomputers [Leighton (1992)]. Also we can represent two -dimensional (2D) indoor spaces with a grid - graph - based model that takes into account the structural and spatial properties of an indoor space. The model developed considers a built environment as a frame of reference at different levels of granularity using a grid-graph-based representation [Skiena (1990)].

2. Basic Concepts

In this paper, graphs are finite and simple, that is, they have no loops or multiple edges. Let G = (V, E) be a graph where V is the vertex set and E is the edge set. A maximal complete sub graph (i.e., clique) is a set of vertices every pair of which are adjacent. A subset D of V (G) is called a dominating set if every vertex in V - D is adjacent to at least one vertex in D. A subset S of V is said to be a minimal dominating set if S - {u} is not a dominating set for any $u \in S$. The domination number γ (G) of G is the smallest size of a dominating set of G. The domination number of its complement $\overline{G}$ is called the co - domination number of G and is denoted by $\gamma(\overline{G})$ or simply $\overline{\gamma}$. A shortest u - v path of a connected graph G is often called a geodesic. The diameter denoted by diam (G) is the length of any longest geodesic. A vertex v of degree zero in G is called an isolated vertex of G.

3. Content of the Paper

The characterizations of interconnection networks are essential in evaluating the performance of the network. We have analyzed the structure of Super Strongly Perfect Graph in Mesh, Torus, Hyper cubes and Grid Networks. We have presented the characterization of Super Strongly Perfect graphs in Mesh, Torus, Hyper cubes and Grid Networks. Also we have investigated the relationship between diameter, domination and co - domination numbers of Mesh, Torus, Hyper cubes and Grid Networks.

3.1. Super Strongly Perfect Graph

A Graph G = (V, E) is Super Strongly Perfect if every induced sub graph H of G possesses a minimal dominating set that meets all the maximal complete sub graphs of H.

Example 1

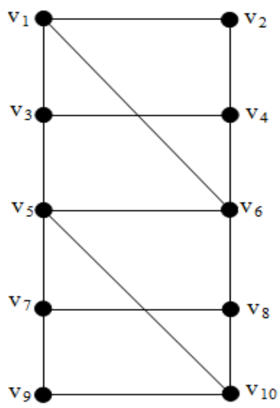

Figure 1: Super Strongly Perfect Graph

Here, $\{v_1, v_4, v_5, v_8, v_9\}$ is a minimal dominating set which meets all maximal cliques of G.

Example 2

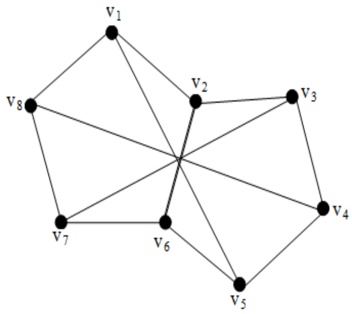

Figure 2: Non - Super Strongly Perfect Graph

Here, {v_1, v_3, v_5, v_7} is a minimal dominating set which does not meet all maximal cliques of G.

4. Cycle

A closed path is called a cycle. A path is a walk in which all vertices are distinct. A walk on a graph is an alternating series of vertices and edges, beginning and ending with a vertex, in which each edge is incident with the vertex immediately preceding it and the vertex immediately following it. An odd cycle is a cycle with odd length, that is, with an odd number of edges. An even cycle is a cycle with even length, that is, with an even number of edges. The number of vertices in a cycle equals the number of edges.

4.1. Theorem [Amutha (2012)]

Let G = (V, E) be a graph with number of vertices n, where $n \geq 5$. If G contains an odd cycle as an induced sub graph, then G is Non - Super Strongly Perfect.

4.2. Theorem [Amutha (2012)]

Let G = (V, E) be a graph with number of vertices n, where $n \geq 5$. Then G is Super Strongly Perfect if and only if it does not contain an odd cycle of length at least 5 as an induced sub graph.

5. Bipartite Graph

A Bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V. A graph is Bipartite if and only if it does not contain an odd cycle [Harary, et, al (1988)]. Therefore, a Bipartite graph cannot contain a clique of size 3 or more. A graph is Bipartite if and only if it is 2-colorable.

Example 3

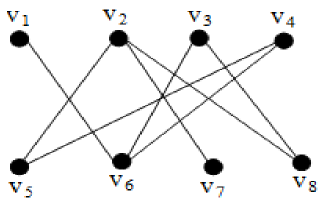

Figure 3: Bipartite graph

5.1. Theorem

Every Bipartite graph is Super Strongly Perfect.

Proof.

Let G be a Bipartite graph.

$\Rightarrow$G does not contain an odd cycle as an induced sub graph.

Now, by the theorem 4.2, G is Super Strongly Perfect.

Hence every Bipartite graph is Super Strongly Perfect. □

5.2. Theorem

Let G be graph with maximal complete sub graph K_2. Then G is Bipartite if and only if it is Super Strongly Perfect.

Proof.

Let G be graph with maximal complete sub graph K_2.

Assume G is Bipartite.
Then by the above theorem G is Super Strongly Perfect.
Conversely assume that G is Super Strongly Perfect with maximal complete sub graph K_2.
To prove G is Bipartite.
Suppose G is non-bipartite with maximal complete sub graph K_2.
Then G contains an odd cycle as an induced sub graph.
Hence by the theorem 4.1, G is Non - Super Strongly Perfect.
Which is a contradiction to the assumption.
Hence G is Bipartite. □

5.3. Theorem

Let G be a Bipartite graph with no isolated vertex which is Super Strongly Perfect, then diam (G) ≥ 3 if and only if $\gamma(\bar{G}) = 2$.

Proof.

Let G be a Bipartite graph which is Super Strongly Perfect.
Assume diam (G) ≥3.
Then there exists at least two vertices u, v in G such that diam (u, v) ≥ 3.
Hence, there does not exist a vertex in G which is adjacent to both u and v.
⇒ All the vertices in $\bar{G}$ are either adjacent to u or v.
⇒ {u, v} is a dominating set of $\bar{G}$.
Since G has no isolated vertex, $\gamma(\bar{G}) \neq 1$.
⇒ $\gamma(\bar{G}) = 2$.
Conversely assume that $\gamma(\bar{G}) = 2$.
Let D = {u, v} be a minimum dominating set of $\bar{G}$.
⇒ All the vertices in $\bar{G}$ are either adjacent to u or v.
Then there does not exist a vertex in G which is adjacent to both u and v.
⇒ diam (u, v) ≥ 3.
⇒ diam (G) ≥ 3.
Hence proved. □

5.4. Theorem

Let G be a Bipartite graph with no isolated vertex which is Super Strongly Perfect then γ (G) >1 if and only if diam ($\bar{G}$) ≤ 3.

Proof.

Let G be a Bipartite graph with no isolated vertex which is Super Strongly Perfect.
Assume diam ($\bar{G}$) ≤ 3.
To prove γ (G) >1.
Suppose γ (G) =1,
⇒ There exists a vertex v ∈ G which is adjacent to all the remaining vertices in G.
⇒ v is an isolated vertex in $\bar{G}$.
⇒ diam ($\bar{G}$) cannot be defined, which is a contradiction to the assumption.
Conversely assume that γ (G) >1,
To prove diam ($\bar{G}$) ≤ 3
Suppose diam ($\bar{G}$) > 3.
Then there exists atleast two vertices u, v in $\bar{G}$ with d (u,v) > 3 in $\bar{G}$.
⇒ $\bar{G}$ has no vertex which is adjacent to both u and v.
All the vertices are either adjacent to u or v in $\bar{G}$.
⇒ {u, v} is a dominating set in G
⇒ γ (G) < 2, which is a contradiction to the assumption.
Hence γ (G) >1. □

5.5. *Theorem*

Let G be a Bipartite graph with no isolated vertex which is Super Strongly Perfect then γ (G) >1 if and only if $\gamma(\bar{G}) = 2$.

Proof.

Let G be a Bipartite graph with no isolated vertex which is Super Strongly Perfect.

Assume $\gamma(\bar{G}) = 2$.

Let D = {u, v} be a minimum dominating set of $\bar{G}$.

⇒ All the vertices are either adjacent to u or v in $\bar{G}$.

⇒ G has no vertex which is adjacent to both u and v.

⇒ γ (G) >1.

Conversely assume that γ (G) >1,

To prove $\gamma(\bar{G}) = 2$.

Since γ (G) >1,

⇒ G has no vertex which is adjacent to all the remaining vertices.

Since G is Bipartite, there exists a bipartition (V_1, V_2) in G, and all the vertices are mutually non adjacent in V_1 and V_2.

⇒ G has no vertex which adjacent to all the remaining vertices.

⇒ In $\bar{G}$, all the vertices in V_1 and V_2 are mutually adjacent and there exists at least one adjacency between any two vertices u, v such that u∈ V_1 , v∈ V_2.

⇒ $\gamma(\bar{G}) = 2$.

Hence proved. □

6. Mesh Network

An n - dimensional mesh M(d_1, d_2...d_n) has {(x_1, x_2...x_n) : $1 \le x_i \le d_i$, $1 \le i \le n$} for its vertex set and vertices (..., x_i, ...) and (..., x_{i+1}, ...), $1 \le i \le n$ are adjacent in M(d_1, d_2...d_n). A mesh is a bipartite graph. If at least one side has even length, the mesh has a hamiltonian cycle. A hamiltonian path exists always. Meshes are not regular, but the degree of any vertex is bounded by 2n. Of course, the degree of a corner vertex is less than the degree of an internal vertex. Therefore, meshes are not vertex symmetric.

The most important mesh-based parallel computers are Intel's Paragon (2-D mesh) and MIT J-Machine (3-D mesh). An n-dimensional toroidal mesh T R(d_1, d_2...d_n) has vertex set same as that of the n-dimensional mesh M(d_1, d_2...d_n). Each vertex (x_1, x_2...x_n) is adjacent to 2n other vertices ($x_1 \pm 1$, x_2...x_n), (x_1, $x_2 \pm 1$...x_n)...(x_1, x_2...$x_n \pm 1$), where additions are performed modulo di ($1 \le i \le n$). [Amutha (2006)].

Example 4

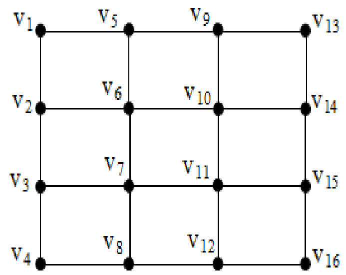

Figure 4: M (4, 4)

6.1. *Theorem*

Every Mesh Network is Super Strongly Perfect.

Proof.

Let G be a Mesh Network.

⇒G does not contain an odd cycle as an induced sub graph.

Now, by the theorem 4.2, G is Super Strongly Perfect.

Hence every Mesh Network is Super Strongly Perfect. □

6.2. Theorem

Let G be a Mesh Network which is Super Strongly Perfect, then diam (G) ≥ 3 if and only if γ ($\bar{G}$) = 2.

Proof.

Let G be a Mesh Network which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.3. □

6.3. Theorem

Let G be a Mesh Network which is Super Strongly Perfect then γ (G) >1 if and only if diam ($\bar{G}$) ≤ 3.

Proof.

Let G be a Mesh Network which is Super Strongly Perfect.

Since G is a Bipartite graph, this theorem is proved by the theorem 5.4. □

6.4. Theorem

Let G be a Mesh Network which is Super Strongly Perfect then γ (G) >1 if and only if γ ($\bar{G}$) = 2.

Proof.

Let G be a Mesh Network which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.5. □

7. Torus Network

A Torus is a mesh with wrap - around links. A 1-dimensional torus is simply a cycle or ring. The torus T R (k, k...k) is called a k - ary n - torus. Tori are bipartite if and only if all side lengths are even. Any torus has a hamiltonian cycle [Amutha (2006)]. Note that the parameters n and m in T R (n, m) designate the side lengths of the network. The Torus T R (n, m) is Bipartite if both m, n are even; otherwise it is non - bipartite.

Example 5

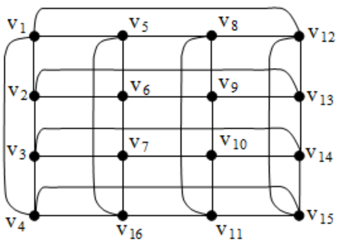

Figure 5: T R (4, 4)

7.1. Theorem

Let G = T R (n, m) be a Torus Network, where n, m ≤ 3, then G is Super Strongly Perfect.

Proof.

Let G = T R (n, m) be a Torus Network, where n, m ≤ 3.

⇒G does not contain an odd cycle as an induced sub graph.

Now, by the theorem 4.2, G is Super Strongly Perfect.

Hence every Torus Network T R (n, m), where n, m ≤ 3, is Super Strongly Perfect. □

7.2. Theorem

Let G = T R (n, m) be a Torus Network, where n, m > 4. If either n or m is odd, then G is Non - Super Strongly Perfect.

Proof.

Let G = T R (n, m) be a Torus Network, where n, m > 4 and either n or m is odd.

⇒G contains an odd cycle as an induced sub graph.

Now, by the theorem 4.1, G is Non - Super Strongly Perfect.

Hence every Torus Network T R (n, m), where n, m > 4 and either n or m is odd, is Non - Super Strongly Perfect. □

7.3. Theorem

Let G = T R (n, m) be a Torus Network, where n, m > 3 and n, m are even, then G is Super Strongly Perfect.

Proof.

Let G = T R (n, m) be a Torus Network, where n, m > 3 and n, m are even.

⇒G does not contain an odd cycle as an induced sub graph.

Now, by the theorem 4.2, G is Super Strongly Perfect.

Hence every Torus Network T R (n, m), where n, m > 3 and n, m are even, is Super Strongly Perfect. □

7.4. Theorem

Let G = T R (n, m) be a Torus Network, where n, m are even, which is Super Strongly Perfect, then diam (G) ≥ 3 if and only if $\gamma (\bar{G}) = 2$.

Proof.

Let G = T R (n, m) be a Torus Network, where n, m are even, which is Super Strongly Perfect.

Since G is a Bipartite graph, this theorem is proved by the theorem 5.3. □

7.5. Theorem

Let G = T R (n, m) be a Torus Network, where n, m are even, which is Super Strongly Perfect then γ (G) >1 if and only if diam $(\bar{G}) \leq 3$.

Proof.

Let G = T R (n, m) be a Torus Network, where n, m are even, which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.4. □

7.6. Theorem

Let G = T R (n, m) be a Torus Network, where n, m are even, which is Super Strongly Perfect then γ (G) >1 if and only if $\gamma (\bar{G}) = 2$.

Proof.

Let G = T R (n, m) be a Torus Network, where n, m are even, which is Super Strongly Perfect.

Since G is a Bipartite graph, this theorem is proved by the theorem 5.5. □

8. Hypercube Network

A Hypercube of order n, denoted by Q_n, is a graph constructed of two copies of the graph Q_{n-1}, where the corresponding nodes of each sub graph are joined. Equivalently, the vertex set V of Q_n consists of all binary sequences of length n on the set {0, 1}. In other words $V = \{x_1x_2 \ldots x_n : x_i \in \{0, 1\}, i = 1, 2 \ldots n\}$. Two vertices $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ are linked by an edge if and only if x and y differ exactly in one coordinate, i.e. $\sum_{i=1}^{n} |x_i - y_i| = 1$. Another definition of Q_n is the Cartesian product of *n* two-vertex complete graphs K_2 [Amutha (2006)]. Every Hypercube is Bipartite.

Example 6

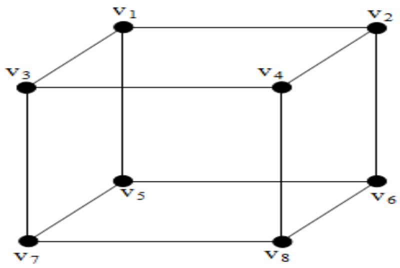

Figure 6: Hypercube Graph - Q_4

8.1. Theorem

Every Hypercube network is Super Strongly Perfect.

Proof.

Let G be a Hypercube network which is Super Strongly Perfect.

⇒G does not contain an odd cycle as an induced sub graph.

Now, by the theorem 4.2, G is Super Strongly Perfect.

Hence every Hypercube network is Super Strongly Perfect. □

8.2. Theorem

Let G be a Hypercube network which is Super Strongly Perfect, then diam (G) ≥ 3 if and only if $\gamma(\bar{G}) = 2$.

Proof.

Let G be a Hypercube Network which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.3. □

8.3. Theorem

Let G be a Hypercube network which is Super Strongly Perfect then γ (G) >1 if and only if diam $(\bar{G}) \leq 3$.

Proof.

Let G be a Hypercube Network which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.4. □

8.4. Theorem

Let G be a Hypercube network which is Super Strongly Perfect then γ (G) >1 if and only if $\gamma(\bar{G}) = 2$.

Proof.

Let G be a Hypercube network which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.5. □

9. Grid Network

A two-dimensional Grid graph $G_{m,n}$ is an m x n graph and it is the graph Cartesian product P_m x P_n of path graphs on m and n vertices, that is $G_{m,n} = P_m \times P_n$. A path graph may also be considered to be a grid graph on the grid n times. A 2x2 grid graph is a 4-cycle. Grid graphs are a special class of planar graphs whose vertices are located on grid points, and whose vertices are adjacent only to their immediate horizontal or vertical neighbors [Xiao and Parhami (2005)]. All Grid graphs are Bipartite.

Example 7

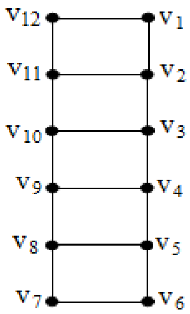

Figure 7: Grid graph - P_6 x P_2

9.1. Theorem

Every Grid graph is Super Strongly Perfect.

Proof.

Let G be a Grid graph which is Super Strongly Perfect.

⇒G does not contain an odd cycle as an induced sub graph.

Now, by the theorem 4.2, G is Super Strongly Perfect.

Hence every Grid graph is Super Strongly Perfect. □

9.2. Theorem

Let G be a Grid graph which is Super Strongly Perfect, then diam (G) ≥ 3 if and only if $\gamma(\bar{G}) = 2$.

Proof.

Let G be a Grid graph which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.3. □

9.3. Theorem

Let G be a Grid graph which is Super Strongly Perfect then γ (G) >1 if and only if diam $(\bar{G}) \leq 3$.

Proof.

Let G be a Grid graph which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.4. □

9.4. Theorem

Let G be a Grid graph which is Super Strongly Perfect then γ (G) >1 if and only if γ ($\bar{G}$) = 2.

Proof.

Let G be a Grid graph which is Super Strongly Perfect

Since G is a Bipartite graph, this theorem is proved by the theorem 5.5. □

10. Conclusion

We have analyzed the structure of Super Strongly Perfect Graph in Mesh, Torus, Hyper cubes and Grid Networks. We have presented the characterization of Super Strongly Perfect graphs in Mesh, Torus, Hyper cubes and Grid Networks. Also we have investigated the relationship between diameter, domination and co - domination numbers of Mesh, Torus, Hyper cubes and Grid Networks. In future, these investigations will be extended to the remaining well known architectures.

11. References

[1] Amutha, A. (2006). *Tree Spanners of Symmetric Interconnection Networks*, Ph.D. thesis, University of Madras.

[2] Amutha, A.; Mary Jeya Jothi, R. (2012). *Characterization of Super Strongly Perfect Graphs in Bipartite Graphs*, Proceedings of an International Conference in Engineering and Business Management (ICMEB 2012), **1**, pp.183 - 185.

[3] Bilardi, G.; Preprata, F. P. (1995). *Horizons of Parallel Computation*, Journal of Parallel and Distributed Computing, **27**, pp. 172 - 182.

[4] Harary, F.; Hayes, J.P.; Wu, H.J. (1988). *A survey of the theory of hypercube graphs*, Comput. Math. Appl., **15**, pp. 277 - 289.

[5] Huang, J.; Palomar, D. P.; Mandayam, N. J.; Walrand, S.; Wicker, B.; Basar, T. (2008). *Game Theory in Communication Systems*, IEEE Journal on Selected Areas in Communications, **26**, pp. 1042 - 1046.

[6] Leighton, F.T. (1992). *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*, Morgan Kaufmann, San Mateo, CA.

[7] Shrinivas, S. G.; et. al., (2010). *Applications of Graph Theory in Computer Science an Overview*, International Journal of Engineering Science and Technology, **2**, pp. 4610 - 4621.

[8] Skiena, S. (1990). *Grid Graph*, In Implementing Discrete Mathematics, Combinatorics and Graph Theory with Mathematica, Addison-Wesley publications, pp. 147-148.

[9] Xiao, W.; Parhami, B. (2005). *Some Mathematical Properties of Cayley Digraphs with Applications to Interconnection Network Design*, Internat. J. Comput. Math. **82**, pp. 521 - 528.