PRE-IDEALS AND NEW TOPOLOGIES USING FUZZY SETS

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Abstract: A fuzzy ideal on a set X is a non empty collection of subsets of X with heredity property which is also closed under arbitrary union and finite intersections. In this paper, we introduce a weak form of fuzzy ideals namely fuzzy pre-ideals and a way to obtain new fuzzy topologies is presented in this paper. Furthermore some interesting results for fuzzy ideals are generalized to fuzzy pre-ideal.

Keywords: compatible ideal$\Im$. F- Compact, $\Im$ - compact, $\Im$C Fuzzy – Compact.

1.INTRODUCTION

Given a non empty set X, a collection $\Im$ of subsets of X is called a fuzzy ideal

If $A \in \Im$ and $B \subseteq A$ implies $B \in \Im$ (heredity)

If $A \in \Im$ and $B \in \Im$ implies $A \cup B \in \Im$ (additivity)

If $X \notin \Im$, then $\Im$ is called a proper ideal.

A fuzzy ideal $\Im$ is called a σ - ideal if the following holds

If $\{A_n : n = 1,2, \ldots\}$ is a countable sub collection of $\Im$, then $\cup\{A_n : n = 1,2, \ldots\} \in \Im$ The notation $(X, \tau, \Im)$ denotes a nonempty set X, a topology τ on X and a fuzzy ideal $\Im$ on X. Given point $x \in X$, $\aleph(x)$ denotes the neighborhood system of x i.e $\aleph(x) = \{ U \in \tau : x \in U \}$. Given a space $(X, \tau, \Im)$ and a subset A of X, we define $A^* (\Im,\tau) = \{x \in X : U \cap A \notin \Im$, for every $U \in \aleph(x)\}$

We simply write A^* for $A^*(\Im, \tau)$, when there are only one ideal $\Im$ and only one topology τ under consideration. If we define cl* on $\wp(X)$ as,$cl^* (A) = A \cup A^*$, for all $A \in \wp(X)$,then cl* is a Kuratowski closure operator. The fuzzy topology determined by this closure operator is denoted by $\tau^* (\Im)$. $\beta (\Im, \tau) = \{U - I : U \in \tau , I \in \Im\}$ is a basis for $\tau^* (\Im)$. The first unified and extensive study on $\tau^* (\Im)$ - topology was done by Jankovic and Hamlett.

Here we introduce a weaker form of ideals, namely fuzzy pre-ideals and a way to obtain new fuzzy topologies from the old fuzzy topologies using pre-ideals..Furthermore some interesting results for fuzzy ideals are generalized to pre-ideals. We shall use cl(A), int(A) to denote closure and interior of a subset A respectively in fuzzy topological space (X, τ) and cl*(A), int*(A) will denotes closure and interior of A respectively with respect to τ^* .

In a fuzzy topological space (X, τ), a subset U is said to be regular- open if int (cl (U)) = U. A subset U in X is regular - closed if cl (int (U)) = U. Clearly U is regular- closed (open) if and only if its complement is regular - open (closed). A subset A of (X, τ) is said to be fuzzy compact if every open cover of A has a finite sub cover. If X is fuzzy compact, then every closed subset of X is also fuzzy compact.

We start with definition of pre-ideals.

Definition 2.1 : A collection P of subsets of X is said to be a fuzzy pre- ideal if

(i) $\phi \in P$

(ii)$A,B \in P \Rightarrow A \cup B \in P$

Examples2.2:

(i) Every fuzzy ideal is a fuzzy pre- ideal.

(ii) If P is the set of all Lebesque measurable sets in R, then P is a fuzzy pre- ideal.(Note that P is not an ideal)

(iii) If P is the set of all compact subsets of a fuzzy topological space(X,τ) then P is a fuzzy pre- ideal not an ideal.

Let (X,τ, P) be a fuzzy topological space with a fuzzy pre- ideal P on X.

We denote $\beta(\tau,P)=\{V-I:V \in \tau, I\in P\}$

Then $\beta(\tau,P)$ is a basis for a fuzzy topology denoted by $\tau^*(P)$ or τ^* on X.

Clearly this topology $\tau^*$is finer than τ.

Theorem 2.3:

Let P_1 and P_2 are fuzzy pre- ideals in X.

1. Then $\tau^*(P_1 \cap P_2,\tau) = \tau^*(P_1,\tau) \cap \tau^*(P_2,\tau)$
2. If $P_1 \cup P_2=\{ I \cup J : I\in P_1$ and $J\in P_2\}$ then $P_1 \cup P_2$ is a pre ideal and $\tau^*(P_1, \tau^*(P_2,\tau))= \tau^*(P_1 \cup P_2,\tau)$.

Proof:

The proof of the theorem is clear.

Theorem 2.4: Let E' denote the derived set of E under $\tau^*(P,\tau)$,where P is a fuzzy pre -ideal Then $\{x : x\in x$ and $x \in G \in \tau \Rightarrow (G \cap E-\{x\}) \notin P\} \subseteq E'$

Remarks 2.5:

(i) $\tau=\tau^*$ if and only if $P \subseteq$ the collection of all fuzzy closed sets in τ .

(ii) Let τ_1 and τ_2 be two fuzzy topologies on X such that τ_1 is weaker than τ_2.

Take P= {collection of all fuzzy closed sets of τ_2}.

Then $\tau_1{}^*(P) = \tau_2$

(iii) Let τ_1 and τ_2 be two fuzzy topologies on X .If P is the collection of all fuzzy closed setsin τ_2 ,then $\tau_1{}^*(p)$ is the fuzzy topology generated by $\tau_1 \cup \tau_2$.

Lemma 2.6:

Let (X,τ) be a fuzzy topological space and P be a fuzzy pre- ideal on X, then $\tau^*=\tau^{**}$

Proof:

Clearly $\tau^* \subseteq \tau^{**}$

Conversely, let V be a τ^{**} neighbourhood of X.

Then there exists $U\in \tau^*$ and $I_1\in P$ such that $x\in U-I_1 \subset V$

Since$U\in \tau^*$,there exists $w\in \tau, I_2\in P$ such that $x\in W-I_2 \subset U$ implies

$x\in W-(I_1 \cup I_2) \subseteq U- I_1 \subseteq V$.

Hence V is τ^* neighbourhood of X.Thus $\tau^*=\tau^{**}$

Lemma 2.7:

Let (X,τ) be a fuzzy topological space and $\Im$ be a fuzzy ideal on X, then $\Im \cap\tau=\{\Phi\}$ if and only if $A^o = \Phi$;for all $A\varepsilon\Im$.

The proof of the lemma is clear. But in the case of Pre-fuzzy ideals the above lemma is not true.

Example 2.8:

Let R be the real line with the usual fuzzy topology and let P be the set of all bounded fuzzy closed subsets of R.i.e.all compact fuzzy subsets of R.Let A $[0,1] \in P$ and $A^o \neq \Phi$;but $P\cap\tau=\{\Phi\}$.

Examples for pre fuzzy ideals P for which $A^o = \Phi$; for all $A\in P$

(i)Let X=R, the real line and τ be the usual fuzzy topology on R. Let P be the collection of all Lebesque measure sets with measure zero.

(ii)Let (X,τ) be any topological space and $\Im_n$ be the fuzzy ideal of nowhere dense sets. If P is a fuzzy pre-ideal such that $P\subseteq\Im_n$, then $A^o = \Phi$; for all $A\in P$

Now we generalize a theorem using this.

Theorem 2.9:

Suppose $A^o = \Phi$; for all $A\in P$, where P is a fuzzy pre-ideal .Let W be τ^* open set and F be its τ^*- closure, then there is τ-open set G such that $W \subset G \subset cl\ G \subset F$.

cl G denotes the τ- closure of G.

Proof:

As β $(P,\tau)=\{V-I:V\in\tau ,I\in P\}$is a basis for τ*and let $W=\cup\{G\alpha-Z\alpha\}$and $X-F = \cup\{ G\beta-Z\beta\}$,where $\{G\alpha\}$ and $\{G\beta\}$are in τ.And $\{Z\alpha\}$ and $\{Z\beta\}$are members of P.

For each $\alpha\in\wedge_1$ and $\beta\in\wedge_2$ we have $\{G\alpha-Z\alpha\} \cap\{G\beta-Z\beta\}=\Phi$.

Thus $G\alpha \cap G\beta \subset Z\alpha\cup Z\beta$.But $Z\alpha\cup Z\beta\in P$ And $G\alpha \cap G\beta$ is τ open

So $G\alpha \cap G\beta \subset (Z\alpha\cup Z\beta)^0=\Phi$.Hence $G\alpha \cap G\beta =\Phi$.If $G = \cup G_\alpha$, it follows that

$G \cap (\cup G_\beta)=\Phi$.Hence cl $G \subset X - \cup G_\beta \subset F$ and Hence $W\subset G \subset$ cl $G\subset F$.

Under the assumption $A^o = \Phi$; for all $A\in P$

We have the following corollaries.

Corollary 2.10:

If (X, τ^*) is hausdroff ,then (X,τ) is hausdroff .

Proof:

Since (X, τ^*) is hausdroff ,for $x\neq y$, there exists τ open sets V_1 and V_2 and

A ,B$\in$P,such that

(i) $x\in V_1$-A and $y\in V_2$-B

(ii) $(V_1-A) \cap (V_2-B) = \Phi$

From (ii) we have $V_1 \cap V_2 \subset A\cup B$,

So by our assumption $V_1 \cap V_2 = \Phi$;

This implies that (X,τ) is hausdroff .

Corollary2.11:

Let (X, τ^*) be a fuzzy topological space.Any continuous function mapping (X, τ^*) into regular space is also continuous if recorded as function from (X,τ) into that space.

Proof:

Let $f: (X, \tau^*) \to(X, \tau_y)$ be continuous.

Fix $x\in X$, $f(x) \in U\subseteq Y$, where U is open in Y.

Find a neighbourhood V of f(x) such that $f(x) \in V\subset$ cl $V\subset U$,Using regular property of Y .

Since $f: (X, \tau^*) \to(X, \tau_y)$ is continuous ,And $f^{-1}(U)$, $f^{-1}(V)$ are τ^* -open

Then f^{-1}(cl V) is τ^* -closedAnd also we have

$f^{-1}(V)\subset$ cl $f^{-1}(V) \subset f^{-1}(clV)\subset f^{-1}(U)$.Then there exists $G\in \tau$,such that

$f^{-1}(V)\subset G\subset$ cl $G\subset$ cl $f^{-1}(V) \subset f^{-1}(clV)\subset f^{-1}(U)$.

So $f(G) \subset U$ i.e given neighbourhood U of f(x),there is a τ neighbourhood G of X,

Such that $f(G) \subset U$ .Hence $f: (X, \tau^*) \to(X, \tau_y)$ is continuous .

Corollary 2.12 :

If (X, τ^*) is regular ,then τ and τ^* coincide.

Proof:

We know that $i_d : (X, \tau^*) \to(X, \tau^*)$ is continuous and hence $i_d : (X, \tau) \to(X, \tau^*)$ is also continuous and hence $\tau= \tau^*$

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