NUMERICAL AND ANALYTICAL SOLUTIONS FOR A NONLINEAR REACTION DIFFUSION SYSTEM

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Abstract - In this paper, we have used two powerful numerical and analytical methods for solving type of partial differential equations called reaction diffusion system in one dimension. In the numerical method, we use implicit method for discretizing the nonlinear reaction tern and implicit method for linear diffusion term. The Tanh method is used to find the analytical solution for this model. The traveling wave solutions are found for this system using the above methods and for generalized logistic growth with nonlinearity of second order. Comparison of two methods show a good agreement.

Keyword: Reaction diffusion system, Tanh method, Finite difference method.

1. Introduction:

Today, the application of Reaction-diffusion systems have been found in many ranges of science, chemical and biological phenomena to medicine (physiology, diseases, etc.), economics, weather prediction, astrophysics and etc[7,3]. There are two main physical aspects in this kind of problems, first aspect is an interaction between two species (or reactions) coupled with a means of transport of their products[1]. One of the essential applications of this model that has been studied widely is heat transfer[2], for example, a study on understanding the candle flame and if there is any similarity with the mechanism of respiration of biological organisms[1]. Also, this model has applications in nerve system[2]. Traveling waves are one of the interest topics and the applications that attract the researcher in their study of reaction diffusion model[5]. In this paper, we study the reaction-diffusion system:

$$\left.\begin{aligned} u_t &= D_u u_{xx} + u(1-\alpha_1 u^n - \alpha_2 w) \\ w_t &= D_w w_{xx} + w(1-\beta_1 w^m - \beta_2 u) \end{aligned}\right\} , \ldots\ldots\ldots(1)$$

Where D_u, D_w are diffusion coefficients, $1-\alpha_1 u^n$ and $1-\beta_1 w^m$ are the generalized logistic growth, α and β are positive constants[9]. We will take n and m as a positive integer focusing on the case $n=m=2$, as we will see that how the problem becomes difficult algebraically to solve when we solve it using both Tanh. When n=m=1, the system (1) return to a simple Lotka-Volterra model[9]. In this paper we focus on solving this model using numerical and analytical methods namely, semi-implicit finite difference[10] as a numerical methods and Tanh method[4,6,8.11] which can use specifically to find the traveling wave solutions for model similar to (1).

2. Finite difference method

Firstly, we discretize (1) using a semi-implicit method as follows:

$$\frac{u_x^{t+\Delta t} - u_x^t}{\Delta t} = D_u\left(\frac{u_{x+\Delta x}^{t+\Delta t} - 2u_x^{t+\Delta t} + u_{x-\Delta x}^{t+\Delta t}}{(\Delta x)^2}\right) + u_x^t(1-\alpha_1(u_x^t)^n - \alpha_2 w_x^t)$$

$$\frac{w_x^{t+\Delta t} - w_x^t}{\Delta t} = D_w\left(\frac{w_{x+\Delta x}^{t+\Delta t} - 2w_x^{t+\Delta t} + w_{x-\Delta x}^{t+\Delta t}}{(\Delta x)^2}\right) + w_x^t(1-\beta_1(w_x^t)^m - \beta_2 u_x^t)$$

Those equations simplify give us:

$$(1+2r_1)u_x^{t+\Delta t} - r_1 u_{x+\Delta x}^{t+\Delta t} - r_1 u_{x-\Delta x}^{t+\Delta t} = u_x^t + \Delta t u_x^t(1-\alpha_1(u_x^t)^n - \alpha_2 w_x^t)$$

$$(1+2r_2)w_x^{t+\Delta t} - r_2 w_{x+\Delta x}^{t+\Delta t} - r_2 w_{x-\Delta x}^{t+\Delta t} = w_x^t + \Delta t w_x^t(1-\beta_1(w_x^t)^m - \beta_2 u_x^t)$$

where, $r_1 = \frac{\Delta t D_u}{(\Delta x)^2}$, $r_2 = \frac{\Delta t D_w}{(\Delta x)^2}$, respectively . The domain of solution 0<x<l is divided to N discrete equally spaced points $x = x_i = (i-l)\Delta x,$ where $i = 1,2,3,4,......., N$ and l is the length of domain. The initial conditions are $u(x,0) = u_0(x)$, $w(x,o) = w_0(x)$. The boundary conditions are no flux or Neumann boundary condition, $u_x = w_x = 0$ at $x = 0, l,$ which are imposed using a three point formula

$$u'(x) = \frac{-3u_x^{t+\Delta t} + 4u_{x+\Delta x}^{t+\Delta t} - u_{x+2\Delta x}^{t+\Delta t}}{2\Delta x} = 0,$$

$$w'(x) = \frac{-3w_x^{t+\Delta t} + 4w_{x+\Delta x}^{t+\Delta t} - w_{x+2\Delta x}^{t+\Delta t}}{2\Delta x} = 0,$$

From discretization we get a system of algebraic equations which can be written in the form:

$$AU^{t+1} = bU^t \text{..............}(2),$$

or

$$\begin{bmatrix} 3 & -4 & 1 & & \\ -r_1 & -r_1 & 1+2r_1 & & \\ \ddots & \ddots & \ddots & & \\ & -r_1 & -r_1 & 1+2r_1 \\ & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} u_1^{t+1} \\ u_2^{t+1} \\ \vdots \\ u_{N-1}^{t+1} \\ u_N^{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ +u_2^t + \Delta t u_2^t (1 - \alpha_1 (u_2^t)^n - \alpha_2 w_2^t) \\ \vdots \\ +u_{N-1}^t + \Delta t u_{N-1}^t (1 - \alpha_1 (u_{N-1}^t)^n - \alpha_2 w_{N-1}^t) \\ 0 \end{bmatrix},$$

and

$$BW^{t+1} = cW^t \text{.........................}(3),$$

$$\begin{bmatrix} 3 & -4 & 1 & & \\ -r_2 & -r_2 & 1+2r_2 & & \\ \ddots & \ddots & \ddots & & \\ & -r_2 & -r_2 & 1+2r_2 \\ & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} w_1^{t+1} \\ w_2^{t+1} \\ \vdots \\ w_{N-1}^{t+1} \\ w_N^{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ +w_2^t + \Delta t w_2^t (1 - \beta_1 (w_2^t)^m - \beta_2 u_2^t) \\ \vdots \\ +w_{N-1}^t + \Delta t w_{N-1}^t (1 - \beta_1 (w_{N-1}^t)^m - \beta_2 u_{N-1}^t) \\ 0 \end{bmatrix}$$

Then, we solve the algebraic system using MATLAB program. For specific values of the parameters and for the initial conditions:

$$u(x,0) = (1 - \tanh(x - l_0))/2,$$

$$w(x,0) = (1 + \tanh(x - l_0))/2.$$

We get nice figures show the traveling wave solutions for this system focusing on the effect of n and m on the propagating of u and w. In other words when we change the degree of the generalized logistic growth in the reaction term, we investigate how the wave of u and w are reacts.

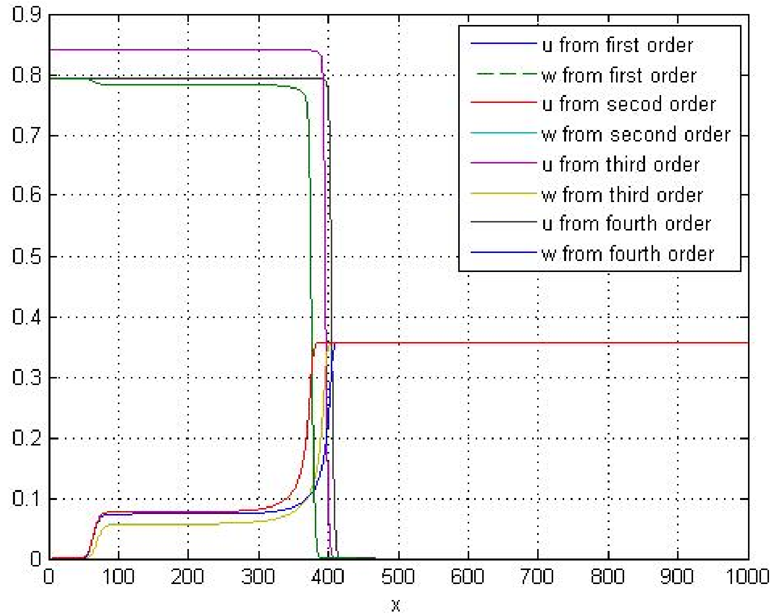

Figure 1: Plot of the numerical solution of (1.1) when $n = 1$, $m = 2$, $n = 2$, $m = 3$, $n = 4$, $m = 2$, $n = 3$, $m = 1$, $n = 3$, $m = 4$ and $D_u = 2.5$, $D_w = 0.5$, $\alpha_1 = 2, \alpha_2 = 0.5, \beta_1 = 1$, $\beta_2 = 2.8$.

We notice that the difference degrees of the term of logistic growth don't make a large effect on the propagation and the shape of the traveling wave solution although there are slightly a difference between these cases.

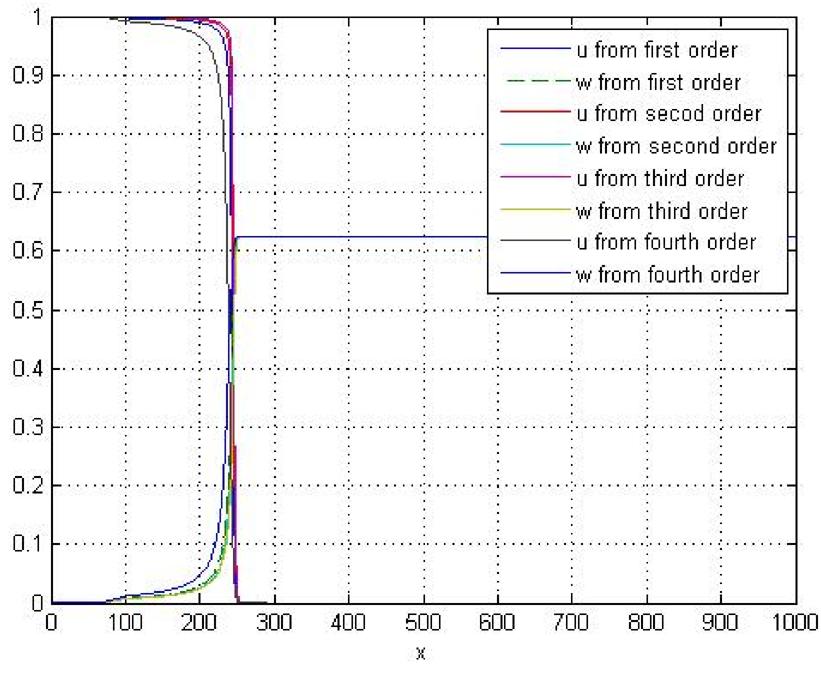

Figure 2: Plot of the numerical solution of eq. (1.1) when $n = 1{,}2{,}3{,}4$, $m = 1$, and $D_u = 1, D_w = 1$, $\alpha_1 = 1, \alpha_2 = 0.7, \beta_1 = 1$, $\beta_2 = 1.6$.

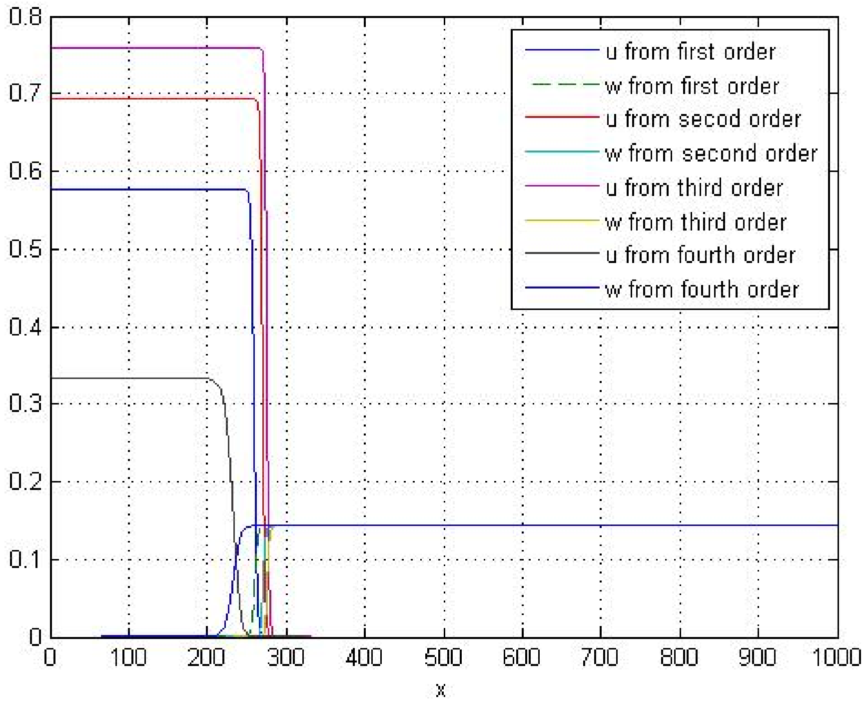

Figure 3: Plot of the numerical solution of (1.1) when $n = 1,2,3,4$, $m = 1$, and $D_u = 2, D_w = 1$, $\alpha_1 = 3, \alpha_2 = 5, \beta_1 = 4$, $\beta_2 = 7$.

3. Tanh method:

Now, we l take the general reaction –diffusion system (1.1) and study its traveling wave solutions using Tanh method when n =m=2. First, we transform (1.1) using

$$u(x,t) = U(\xi) \text{ and } w(x,t) = W(\xi).......(4),$$

where, $\xi = k(x - \lambda t)$,

to get

$$\left.\begin{array}{l} -k\lambda \dfrac{dU}{d\xi} - D_u k^2 \dfrac{d^2U}{d\xi} - U(1-\alpha_1 U^2 - \alpha_2 W) = 0 \\ -k\lambda \dfrac{dW}{d\xi} - D_w k^2 \dfrac{d^2W}{d\xi} - W(1-\beta_1 W - \beta_2 U) = 0 \end{array}\right\}, \qquad (5)$$

where

$$\left.\begin{array}{l} \dfrac{d}{d\xi} = (1-Y^2)\dfrac{d}{dY} \\ \dfrac{d^2}{d\xi^2} = -2Y(1-Y^2)\dfrac{d}{dY} + (1-Y^2)^2 \dfrac{d^2}{dY^2} \\ \dfrac{d^3}{d\xi^3} = +2(1-Y^2)(3Y^2-1)\dfrac{d}{dY} - 6Y(1-Y^2)^2 \dfrac{d^2}{dY^2} + (1-Y^2)^3 \dfrac{d^3}{dY^3} \end{array}\right\}, \qquad (6)$$

We introduce $Y = \tanh(\xi)$ and replace eq. (6) by

$$\left.\begin{array}{l} -k\lambda(1-Y^2)\dfrac{dF(Y)}{dY}-D_uK^2(-2Y(1-Y^2)\dfrac{dF(\gamma)}{d\gamma}+(1-Y^2)^2\dfrac{d^2F(\gamma)}{d\gamma}) \\ -F(Y)(1-\alpha_1(F(Y))^2-\alpha_2G(Y))=0 \\ -K\lambda(1-Y^2)\dfrac{dG(Y)}{dY}-D_wK^2(-2Y(1-Y^2)\dfrac{dG(Y)}{d(Y)}+(1-Y^2)^2\dfrac{d^2G(Y)}{dY}) \\ -G(Y)(1-\beta_1(G(Y))^2-\beta_2F(Y))=0 \end{array}\right\}, \qquad (7)$$

Now, to determine the parameter N and M we balance the linear term of highest order with highest order of nonlinear terms, in (7). The balance of U'' with U^3 obtains N+2 = 3N, and therefore N = 1 where $[U''=\dfrac{d^2F(Y)}{dY^2}]$. Similar balancing of W'' with W^3 gives M + 2= 3M, then M=1, where $[W''=\dfrac{d^2G(Y)}{dY^2}]$.

Now, the Tanh method admits the use of the finite expansion for :

$$\left.\begin{array}{l} u(x,t)=F(Y)=ao+a_1Y \\ w(x,t)=G(Y)=bo+b_1Y \end{array}\right\}, \qquad (8)$$

where, $a_1\neq 0, b_1\neq 0$. Substituting (8) in (7) we get,

$$-k\lambda(1-Y^2)(a_1)-D_uk^2[-2Y(1-Y^2)(a_1)+(1-Y^2)^2(0)]-(ao+a_1Y)(1-\alpha_1$$

$$(a_0^2+2a_0a_1Y+a_1^2Y^2)-\alpha_2(b_0+b_1Y))=0, \qquad (9)$$

and

$$-k\lambda(1-Y^2)(b_1)-D_wk^2[-2Y(1-Y^2)(b_1)+(1-Y^2)^2(0)]-(b_0+b_1Y)(1-\beta_1$$

$$(b_0^2+2b_0b_1Y+b_1^2Y^2)-\beta_2(a_0+a_1Y))=0, \qquad (10)$$

then eq.(9) becomes,

$$(-a_0+\alpha_1a_0^3+\alpha_2a_0b_0-k\lambda a_1)Y^0+(-a_1+3a_1\alpha_1a_0^2+\alpha_2a_0\beta_1$$

$$+\alpha_2a_1b_0+2D_uk^2a_1)Y^1+(3\alpha_1a_0a_1^2+\alpha_2a_1b_1+k\lambda a_1)Y^2$$

$$+(\alpha_1a_1^3-2D_uk^2a_1)Y^3=0, \qquad (11)$$

and this leads to:

$$\left.\begin{aligned}
&Y^0 : (-a_0 + \alpha_1 a_0^3 + \alpha_2 a_0 b_0 - k\lambda a_1) = 0\\
&Y^1 : (-a_1 + 3\alpha_1 a_1 a_0^2 + \alpha_2 a_0 b_1 + \alpha_2 a_1 b_0 + 2D_u k^2 a_1) = 0\\
&Y^2 : (3\alpha_1 a_0 a_1^2 + \alpha_2 a_1 b_1 + k\lambda a_1) = 0\\
&Y^3 : (\alpha_1 a_1^3 - 2D_u k^2 a_1) = 0
\end{aligned}\right\}, \qquad (12)$$

Similarly eq.(10) becomes:

$$(-b_0 + \beta_1 b_0^3 + \beta_2 b_0 a_0 - k\lambda b_1)Y^0 + (-b_1 + 3\beta_1 b_1 b_0^2 + \beta_2 b_0 a_1$$

$$+ \beta_2 b_1 a_0 + 2D_w k^2 b_1)Y^1 + (3\beta_1 b_0 b_1^2 + \beta_2 b_1 a_1 + k\lambda b_1)Y^2$$

$$+ (\beta_1 b_1^3 - 2D_w k^2 b_1)Y^3 = 0, \qquad (13)$$

and this leads to:

$$\left.\begin{aligned}
&Y^0 : (-b_0 + \beta_1 b_0^3 + \beta_2 b_0 a_0 - k\lambda b_1) = 0\\
&Y^1 : (-b_1 + 3\beta_1 b_1 b_0^2 + \beta_2 b_0 a_1 + \beta_2 b_1 a_0 + 2D_w k^2 b_1) = 0\\
&Y^2 : (3\beta_1 b_0 b_1^2 + \beta_2 b_1 a_1 + k\lambda b_1) = 0\\
&Y^3 : (\beta_1 b_1^3 - 2D_w k^2 b_1) = 0
\end{aligned}\right\}, \qquad (14)$$

The remaining constants are easily found through simple algebra such that

$$a_1 = \pm k\sqrt{\frac{2D_u}{\alpha_1}}\ ,\ b_1 = \pm k\sqrt{\frac{2D_w}{\beta_1}},$$

$$a_0 = \pm(2\alpha_2\sqrt{\frac{D_u D_w}{\alpha_1 \beta_1}} + \lambda\sqrt{\frac{2D_u}{\alpha_1}})/6D_u,\ b_0 = \pm\frac{2D_u k^2}{\alpha_2}$$

Finally, we find the solution in the form

$$u(x,t) = a_0 + a_1 \tanh k(x - \lambda t)$$

$$w(x,t) = b_0 + b_1 \tanh k(x - \lambda t)$$

The traveling wave solutions of u(x,t) and w(x,t) can be plot using by MATLAB as shown in figure 4.

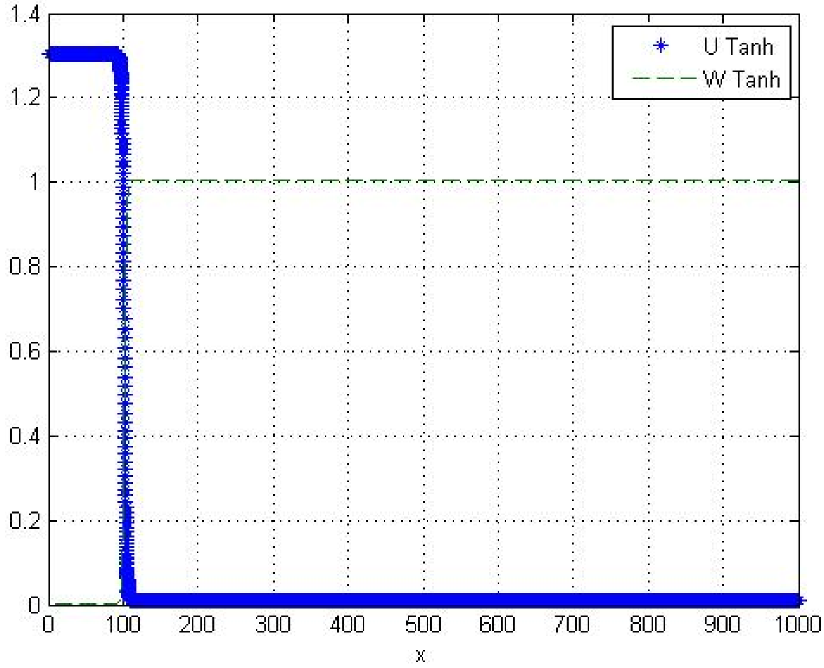

Figure 4: plot the numerical solution of the reaction–diffusion system (1.1) by the Tanh method when:

$$a_1 = 0.6, a_2 = 0.5, b_1 = 1, b_2 = 0.7, D_u = 1, D_w = 1, k = 0.355, t = 70$$

Now, the comparison of the solution of the reaction diffusion system (1.1) using the finite difference method and the Tanh method when (m=n=2) shows a good agreement between them as shown in the figure (5).

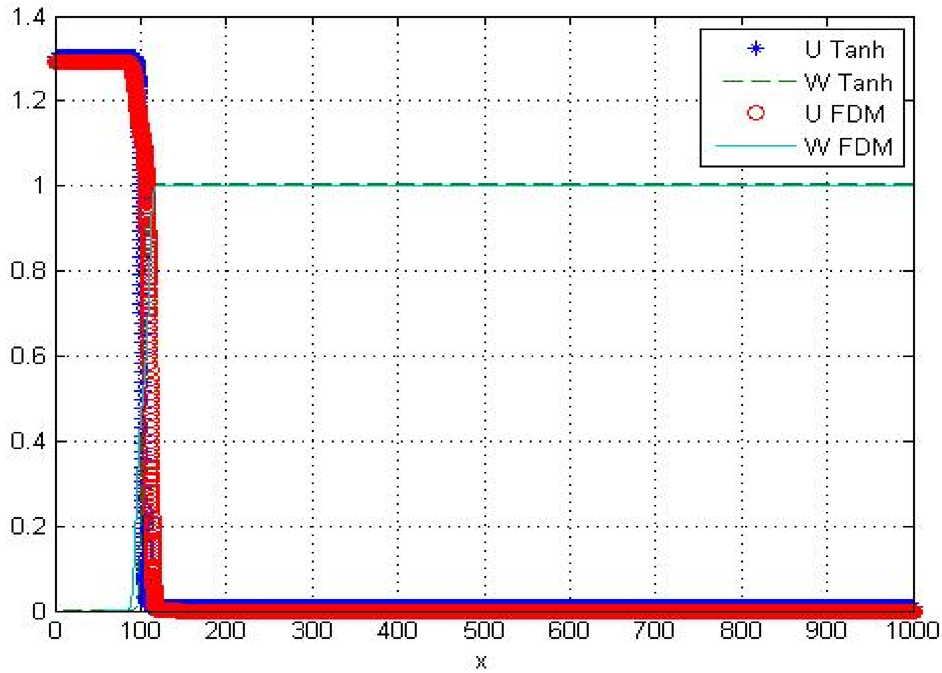

Figure 5 : plot the comparison for eq. (1) between Tanh method and finite difference method when (n=2 , m=2) and

$$a_1 = 0.6, a_2 = 0.5, b_1 = 1, b_2 = 0.7, D_u = 1, D_w = 1, k = 0.355, t = 70$$

4. Conclusion

In this paper, we study a reaction diffusion system (1.1) with nonlinear and generalized logistic growth. Traveling wave solutions for this model are found using finite difference and Tanh methods. The focus on the nonlinearity of generalized logistic growth was in the second order, where the first order will return the system to a Lotka-Volterra model. Both numerical and analytical methods are sufficient to find the traveling wave solution for this model.

5. References

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